CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA

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1. Introduction. The concept of the conditional expectation in probability theory is very important, especially fundamental for the martingale theory. In a book [1][1] of J. Doob, various properties of the conditional expectation in a probability space are described for the random variables having the expectations. While in a recent paper [2], Shuh-teh C. Moy has discussed the characteristic properties of the conditional expectation as a linear transformation of the space of all extended real-valued measurable functions on a probability space into itself.

The present paper deals with the conditional expectation as a mapping of a space of measurable operators belonging to a L^1-integrable class associated with a certain W^*-algebra into itself. This generalization seems to be a first attempt of a non-commutative probability theory. The non-commutative integration theory[2] of I. E. Segal (cf. [3]) has its due application in the subject.

We shall show in § 2, the existence of the conditional expectation for the space of measurable operators of the L^1-integrable class associated with a certain W^*-algebra, and in § 3, the uniqueness in a certain sense of such a mapping which is a generalization of a characterization theorem of S. C. Moy.

2. Existence of conditional expectation. Let A be a W^*-algebra, acting on a Hilbert space H, with a complete (faithful) normal trace μ with $\mu(1) = 1$.

Let A_1 be an arbitrary (but fixed) W^*-subalgebra of A. In this section we shall introduce a conditional expectation in A relative to A_1.

First we shall prove in $L^1(A)$ the existence theorem of conditional expectation where $L^1(A)$ consists of all integrable operators on H with respect to the L^1-norm $\|x\|_1 = \mu(|x|)$ (cf. [3] Def. 3. 2, Cor. 10. 1 and Cor. 11. 3) which are associated with the W^*-algebra A. Similarly we denote the space $L^1(A_1)$ associated with the W^*-subalgebra A_1, then $L^1(A_1)$ can be considered as a closed subspace of $L^1(A)$.

THEOREM 1.[3] *There exists a mapping $x \to x^e$ from $L^1(A)$ onto $L^1(A_1)$ satisfying the following conditions: for any $x, y \in L^1(A)$ and any complex numbers α, β*

$$\text{(i)} \qquad (\alpha x + \beta y)^e = \alpha x^e + \beta y^e,$$

1) Numbers in brackets refer to the reference at the end of the paper.

2) J. Dixmier has also described the similar theory under a different way (cf. [4]). In the present paper, we shall use the definitions and terminologies of I. E. Segal (cf. [3]). We shall denote the product, sum and difference of measurable operators x, y merely by $xy, x+y$ and $x-y$, e. g., ey implies $x \cdot y$ in the notations in [3]. When $x=y$ nearly everywhere, we shall denote merely $x=y$ (n. e.) or $x=y$.

3) After we had proved the Thm 1, we have been pointed out by M. Nakamura that the existence of mapping $x \to x^e$ from A to A_1 was proved by Dixmier using his operator method (cf. Thm. 8 of [4]). In this paper, we shall prove Thm. 1 by Radon-Nikodym Thm. of Segal (cf. [3]) and extend it onto $L^1(A)$.