ON THE ASPHERICITY OF THE HIGHER DIMENSIONAL COMPLEX

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1. W. H. Cockcroft [1] discussed the non-asphericity of the two-dimensional complex K, which is composed of a given non-aspherical two-dimensional complex L, and two-dimensional cells attached to L. According to his consequences, if $\pi_1(L)$ is (i) Abelian or (ii) a finite group or (iii) a free group, or if L contains only one two-dimensional cell, then K is non-aspherical.

In the present note, we consider the n-dimensional complex K $(n \geqq 3)$, which is composed of a given non-aspherical n-dimensional complex L, and n-dimensional cells attached to it. In this case, we shall prove the asphericity of K in the complete form; namely, K is aspherical, if and only if $\pi_r(L) = 0$ for $1 < r < n-1$ when $n \geqq 4$, $\pi_{n-1}(L)$ is a non-zero free $\pi_1(L)$-module and $H_n(\widetilde{L}) = 0$, where $\widetilde{L}$ is the universal covering complex L of. Then, it is shown that $\widetilde{L}$ is of the same homotopy type as a set of $(n-1)$-spheres having a point in common.

2. Let L be a connected, n-dimensional CW-complex [3] $(n \geqq 3)$. We shall say, following Hurewicz [2], that L is aspherical, if and only if its homotopy groups satisfy the conditions

$$\pi_r(L) = 0 \qquad (r > 1). \tag{2.1}$$

LEMMA (2.2). *L is aspherical, if and only if*

$$\pi_r(L) = 0 \qquad (1 < r \leqq n). \tag{2.3}$$

In fact, we need only to show the sufficiency. From (2.3), we obtain, using the Hurewicz' theorem,

$$\pi_{n+1}(L) \approx \pi_{n+1}(\widetilde{L}) \approx H_{n+1}(\widetilde{L}) = 0,$$

where $\widetilde{L}$ is the universal covering complex of L, and $H_{n+1}(\widetilde{L})$ is its integral homology group. Using the same arguments as above, we get inductively (2.1) for every $r > n$.

Next, let K be a complex such that

$$K = L \cup \{e_i^n\} \tag{2.4}$$

where $\{e_i^n\}$ is a set of n-cells attached to the $(n-1)$-skeleton of L.

LEMMA (2.5). *If L is a non-aspherical complex such that $n \geqq 4$, and if*

$$\pi_r(L) \neq 0 \qquad (1 < r < n-1),$$

for at least one r, then K is non-aspherical.

In fact, we can easily see the non-asphericity of K from a part of the exact homotopy sequence