CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA, II

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1. Introduction. The theory of integration on a measure space has been generalized to a W^*-algebra by Segal [10] and Dixmier [2] as a non-commutative extension of it. Applying their theory, some parts of the probability theory may be described in a certain W^*-algebra. In the paper of Dixmier [2], he has proved the existence of a mapping $x \rightarrow x^e$ defined on a semi-finite W^*-algebra A acting on a Hilbert space H into its W^*-subalgebra A_1 with the similar properties of the Dixmier's trace (= natural mapping) in the finite W^*-algebra, A being semi-finite provided every non-zero projection in A contains a non-zero finite projection in A (cf. [5]). In the previous paper [11], we have discussed for a σ-finite finite W^*-algebra A (with the faithful normal trace μ with $\mu(I) = 1$) that the mapping $x \rightarrow x^e$ is defined on $L^1(A)$ and valued on $L^1(A_1)$ and it has the likewise properties with the conditional expectation in the usual probability space, and we have also called it the conditional expectation relative to the W^*-subalgebra A_1, where $L^1(A)$ being a Banach space of all integrable operators on H in the sense of Segal (cf. [10]) which coincides with that in the sense of Dixmier (cf. [2]) as Banach space. Nakamura-Turumaru have also given a very simple proof of the characterization theorem of the conditional expectation in A (cf. [8]).

If A is a commutative W^*-algebra with a faithful normal trace μ, then there exists a probability space $(\Omega, \mathbf{B}, \nu)$ such that, considering the space B of all bounded random variables as the multiplication algebra on a Hilbert space $L^2(\Omega, \mathbf{B}, \nu)$, B is isomorphic with A by the canonical mapping ϕ satisfying $\mu(x) = \int_\Omega (\phi^{-1}(x))(\omega)d\nu(\omega)$ for every $x \in A$. Conversely, let $(\Omega, \mathbf{B}, \nu)$ be a probability space. Then the multiplication algebra B is a W^*-algebra on $L^2(\Omega, \mathbf{B}, \nu)$ and μ, defined by the above equation, is a faithful normal trace on it. Furthermore, the canonical mapping ϕ defines an isomorphism between $L^r(A)$ and $L^r(\Omega, \mathbf{B}, \nu)$ as Banach spaces ($r \geqq 1$), $L^r(A)$ being the Banach space defined by Dixmier (cf. [2]). For any W^*-subalgebra A_1 of A, there corresponds a σ-subfield $\mathbf{B}_1$ of $\mathbf{B}$, and A_1, $L^r(A_1)$ are isomorphic with B_1, $L^r(\Omega, \mathbf{B}_1, \nu)$ respectively, where B_1 being the multiplication algebra of the bounded random variables on $(\Omega, \mathbf{B}_1, \nu)$. The conditional expectation defined for the commutative algebra A (relative to the A_1) is transformed to the one defined for the corresponding probability space $(\Omega, \mathbf{B}, \nu)$ (relative to the $\mathbf{B}_1$) by the canonical mapping (cf. [7] and [11]).

In the probability theory, the martingales have been investigated by many authors, particularly by Doob, Lévy and Ville (cf. [3]), which is defined