A GENERAL APPROXIMATION METHOD OF EVALUATING MULTIPLE INTEGRALS

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The object of this paper is to investigate a general method concerning the approximate evaluation of multiple integrals with periodic continuous functions as integrands. As it will be shown, our general method has some practical advantage when applied especially to the cases of double integrals and triple integrals which are taken over circular regions and spherical domains respectively.

Throughout the paper we always denote, for a continuous function $f(x_1, \dots, x_n)$ defined on a certain n-dimensional domain D,

$$M_f = \max_D |f(x_1, \dots, x_n)|,$$

$$\omega_f(\delta_1, \dots, \delta_n) = \max_D |f(x_1, \dots, x_n) - f(x_1', \dots, x_n')|, \quad (|x_i - x_i'| \leqq \delta_i)$$

where the maxima are all taken with respect to $(x_1, \dots, x_n) \in D$, $(x_1', \dots, x_n') \in D$, the letter one being restricted to $|x_i - x_i'| \leqq \delta_i$ $(1 \leqq i \leqq n)$. Moreover, for a real number $x > 0$, we shall always denote its decimal part by $<x>$, i.e. $<x> = x - [x]$, $[x]$ being the integral part of x.

1. A fundamental lemma and its consequences. Hereafter D_n is always used to denote an n-dimensional hypercubic domain of $(x_1, \dots, x_n)$ in euclidean n-space, namely, D_n; $0 \leqq x_1 \leqq 1, \dots, 0 \leqq x_n \leqq 1$. We are now going to establish a useful lemma which actually forms a basis of our method.

LEMMA. *Let $f(x_1, \dots, x_k, y_1, \dots, y_k)$ be any continuous function defined on D_{2k}. Then for all positive integers $N_i \geqq 2$ $(i = 1, \dots, k)$ we have*

$$(1) \qquad \left| \int_{D_{2k}} f(x, y)\, dx\, dy - \int_{D_k} f(x, <Nx>)\, dx \right| \leqq 2\omega_f\left(\frac{1}{N_1}, \dots, \frac{1}{N_k}, 0, \dots 0, \right),$$

where x, dx etc. are abbreviations for $(x_1, \dots, x_k)$, $dx_1 \dots dx_k$ etc. respectively; and $<Nx>$ stands for $<N_1x_1>, \dots, <N_kx_k>$.

As is easily seen, the meaning of this lemma is that it replaces the $2k$-fold integral by the k-fold integral with an error estimation expressed by the modulus of continuity $\omega_f(N_1^{-1}, \dots, N_k^{-1}, 0, \dots, 0)$. In order to save space in its proof, we have to adopt more abbreviations here, e.g.

$$\int_{(\nu-1)x}^{\nu\Delta x} \text{ stands for} \int_{(\nu_1-1)\Delta x_1}^{\nu_1\Delta x_1} \cdots \int_{(\nu_k-1)\Delta x_k}^{\nu_k\Delta x_k}; \quad \sum_\nu \text{ stands for} \sum_{\nu_1=1}^{N_1} \cdots \sum_{\nu_k=1}^{N_k};$$