RIEMANN-CESÀRO METHODS OF SUMMABILITY II*)

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1. Introduction. In the previous paper [3], we defined Riemann-Cesàro method of summability which includes well-known Riemann methods of summability (R, p) and (R_p). In this paper, we shall consider some Tauberian theorems for this summability.

In terms of standard notations used by Zygmund [10; p. 42] and others, Cesàro transform of order α of $\sum a_n$ is defined by $\sigma_n^\alpha = s_n^\alpha / A_n^\alpha$ where s_n^α and A_n^α are given by the relations

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1} \text{ and } \sum_{n=0}^{\infty} s_n^\alpha x^n = \frac{\sum_{n=0}^{\infty} s_n x^n}{(1-x)^\alpha} = \frac{\sum_{n=0}^{\infty} a_n x^n}{(1-x)^{\alpha+1}}.$$

It is well-known that $A_n^\alpha \sim n^\alpha / \Gamma(\alpha+1)$, $\alpha \neq -1, -2, \dots,$ as $n \to \infty$. A series is said to be evaluable (C, α) to s if $\sigma_n^\alpha \to s$ as $n \to \infty$. In the following, let α be a real number, not necessarily an integer, for which $\alpha \geqq -1$ and let p be a positive integer. A series $\sum_{n=1}^{\infty} a_n$ is said to be evaluable to zero by Riemann-Cesàro method of order p and index α, or briefly, to be evaluable (R, p, α) to zero, if the series

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^\alpha \left(\frac{\sin nt}{nt}\right)^p$$

converges in some interval $0 < t < t_0$ and its sum tends to zero as $t \to 0$. Under these definitions, summability $(R, p, -1)$ and $(R, p, 0)$ is reduced to summability (R, p) and (R_p), respectively. It is known [3] that summability (R, p, α), when $-1 \leqq \alpha < p-1$ and $p \geqq 2$, is regular, or more precisely, summability $(C, p-1-\delta)$, $0 < \delta < 1$, implies summability (R, p, α) when $-1 \leqq \alpha < p - 1 - \delta$, while summability $(R, 1, \alpha)$ is not regular when $-1 \leqq \alpha \leqq 0$.

Concerning summability (R, p), Kanno [5] proved the following

THEOREM K. *Let p be a positive integer. Suppose that*

$$s_n^\beta = o(n^\gamma), \tag{1.1}$$

when $0 < \gamma < \beta$, *and*

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}), \tag{1.2}$$

*) This paper is a continuation of the previous paper [3]. Cf. R. P. Agnew, Properties of generalized definitions of limit, Bull. Amer. Math. Soc., 45 (1939), 689–730.