# ON THE RIEMANN-CESÀRO SUMMABILITY OF SERIES AND INTEGRALS

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**1. Introduction and notation.** Otto Szász ([5], p. 1139, Theorem 4 and p. 1223, Theorem 1) has proved Tauberian theorems for series, involving the passage from Abel or (A) summability to each of the Riemann summabilities (R, 1) and ($R_1$), included in the statement:

THEOREM A. *If* $\sum_{k=1}^{\infty} a_k$ *is summable* (A) *to a finite value* $l$, *and*

$$(T_A) \qquad \text{either } \sum_{k=1}^{n} k|a_k| = O(n), \text{ or } \sum_{k=n}^{2n} (|a_k| - a_k) = O(1), \quad n \to \infty,$$

*then* $\sum_{k=1}^{\infty} a_k$ *is summable* (R, 1) *to* $l$ *and also summable* ($R_1$) *to* $l$.

It is known ([5], p. 1139, Lemma 1) that the second alternative of condition ($T_A$) along with the summability (A) of $\Sigma a_k$ implies the first alternative of ($T_A$); and so Theorem A need be stated with only the first alternative of ($T_A$) which Szász uses in his proof of Theorem A without however, explicitly mentioning it as an alternative hypothesis. The main object of this paper is to establish two results: (i) Theorem I'(A) at the end, which is a generalization of Theorem A with the first alternative of hypothesis ($T_A$), for the Riemann-Cesàro summability ($R, p, \alpha$) recently defined by Hirokawa ([2], § 1) whose case $p = 1$, $\alpha = -1$ is summability (R, 1) and case $p = 1$, $\alpha = 0$ is summabity ($R_1$), (ii) an integral analogue of Theorem I' (*A*) stated as Theorem I (A) in the last section.[1]

The notation and the definitions used in Theorem I(A) and other integral theorems are as follows. For a real function $a(u)$ bounded and integrable[2]

1) It must be borne in mind that the parallelism between series and integrals is destroyed to some extent by instances of theorems for series, such as the limitation theorem for series summable ($C, \alpha$), $\alpha > -1$ ([1], Theorem46), which have no integral analogues. Thus one of Hirokawa's general theorems ([2], Theorem 3) has no integral analogue which can be proved by his method since it depends on the limitation theorem referred to. On the other hand, a theorem for integrals, such as Theorem I (A) of this paper, may present additional complications when we try to adapt its proof to obtain its analogue for series. It may be added here that analogous theorems or formulae for integrals and for series, wherever they occur in this paper, bear the same number, unaccented (e.g. I, 1 etc.) or accented (e.g. I', 1' etc.), according as the theorems or the formulae are for integrals or for series.

2) As in Hardy [1], integrability is in the Lebesgue sense and every integral $\int_0^\infty$ is defined in the Cauchy-Lebesgue sense as $\lim_{x\to\infty} \int_0^x$.