ON THE PROJECTION OF NORM ONE IN W*-ALGEBRAS II

JUN TOMIYAMA

(Received June 20, 1958)

In this paper, we shall study the projection of norm one in W^*-algebras following [7]. Firstly, we obtain the general decomposition theorem of a projection of norm one π from a W^*-algebra $\mathbf{M}$ to its C^*-subalgebra $\mathbf{N}$ showing that $\mathbf{N}$ is decomposed into the maximal W^*-representable direct summand and the rest. Restricting ourself to the case of $\mathbf{N}$ being a W^*-representable $*$-subalgebra, we prove that π is decomposed into three parts by three orthogonal central projections z_1, z_2, z_3 of $\mathbf{N}$. The first component is a normal projection of norm one from $\mathbf{M}$ to $\mathbf{N}z_1$, the second singular one to $\mathbf{N}z_2$ and z_1, z_2 are maximal central projections having these properties. In the last section we discuss on the σ-weak continuity property of π and the relation to the other continuity. We can prove that π is σ-weakly continuous if and only if the kernel of π is σ-weakly closed.

1. Preliminaries. Consider a W^*-algebra $\mathbf{M}$, its conjugate space $\mathbf{M}^*$ and the space $\mathbf{M}_*$ of all σ-weakly continuous linear functionals on $\mathbf{M}$. We define the operators R_a and L_a on $\mathbf{M}^*$ for each $a \in \mathbf{M}$ such that

$$< x, R_a\varphi > = < xa, \varphi > \text{ and } < x, L_a\varphi > = < ax, \varphi >$$

for all $a \in \mathbf{M}$, $\varphi \in \mathbf{M}^*$. The following properties are easily verified: $R_{(\lambda a+\mu b)} = \lambda R_a + \mu R_b$, $L_{(\lambda a+\mu b)} = \lambda L_a + \mu L_b$, $R_{ab} = R_a R_b, L_{ab} = L_b L_a$, where a and b are arbitrary elements of $\mathbf{M}$ and λ, μ complex numbers.

A subspace of $\mathbf{M}^*$ which is invariant both for every R_a and every L_a is called an invariant subspace. It can be shown that there exists a one-to-one correspondence between the σ-weakly closed ideal m of $\mathbf{M}$ and the closed invariant subspace V of $\mathbf{M}_*$ such that $m = V^0$ and $V = m^0$ where V^0 and m^0 denote the polar of V and m in $\mathbf{M}$ and $\mathbf{M}_*$ respectively.

A positive linear functional φ is called singular if there exists no non-zero positive normal linear functional ψ such as $\psi \leqq \varphi$; we denote the closed subspace generated by all singular linear functionals on $\mathbf{M}$ by $\mathbf{M}_*^{\perp}$. $\mathbf{M}_*^{\perp}$ is an invariant subspace of $\mathbf{M}^*$. It can be shown that any closed invariant subspace V is decomposed such as

$$V = (V \cap \mathbf{M}_*) \underset{l^1}{\oplus} (V \cap \mathbf{M}_*^{\perp}), \text{ in particular } \mathbf{M}^* = \mathbf{M}_* \underset{l^1}{\oplus} \mathbf{M}_*^{\perp},$$

the sum being l^1-direct sum.

A uniformly continuous linear homomorphism π from a W^*-algebra $\mathbf{M}$ to a W^*-algebra $\mathbf{N}$ is called singular if ${}^t\pi(\mathbf{N}_*) \subset \mathbf{M}_*^{\perp}$, where ${}^t\pi$ denote the transpose of π. We can prove that a positive singular mapping π from $\mathbf{M}$ to $\mathbf{N}$ has the property that there exists no non-zero normal linear homo-