ON THE DIFFERENTIAL GEOMETRY OF TANGENT BUNDLES OF RIEMANNIAN MANIFOLDS

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Introduction. Let M^n be an n-dimensional differentiable manifold. The set of all tangent vectors of M^n constitutes, with a natural topology, the so-called tangent bundle of M^n. We denote it by $T(M^n)$. The set of all unit vectors of M^n constitutes a subbundle of $T(M^n)$ and is a sphere-bundle over M^n. It is called the tangent sphere-bundle of M^n. We shall denote it by $T_1(M^n)$.

H. Poincaré used the tangent sphere-bundles of ovaloids in three dimensional Euclidean space, i. e. the phase spaces of the ovaloids, to prove the existence of certain closed geodesics on the ovaloids. He introduced a Riemannian metric on the tangent sphere-bundles and considered the geodesic flow on it. As the metric of tangent bundles of Riemannian manifolds seems to be important, we would like to study differential geometry of tangent bundles of Riemannian manifolds by introducing on it natural Riemannian metrics. In this papar we shall do it by restricting ourselves only to the tangent bundles $T(M^n)$.

1. Incompressible vector fields over a Riemannian manifold M^n. Let ξ be a differentiable vector field over a differentiable Riemannian manifold M^n and suppose its components with respect to an arbitrary coordinate neighborhood U be ξ^i.[1] The following Lemma is well-known:

LEMMA 1. *In order that the infinitesimal transformation $Xf = \xi^i \dfrac{\partial f}{\partial x^i}$ leaves invariant volume element of the Riemannian manifold M^n, it is necessary and sufficient that the divergence*

$$\xi^i_{,i} = 0 \tag{1.1}$$

identically, where the comma in $\xi^i_{,i}$ means the covariant derivative.

If the divergence of a vector field ξ over M^n vanishes identically, it is clear that the one parameter group of transformations of M^n generated by $Xf = \xi^i \dfrac{\partial f}{\partial x^i}$ is a group whose elements are homeomorphisms which leave invariant volumes of all domains in M^n. On account of this fact we shall say that ξ is an *incompressible vector field* if the divergence of the vector field

1) We assume that groups of indices located in the left hand side of the following lines take values which lie on the right hand side of the corresponding lines.

$$\left.\begin{matrix} h, i, j, k, l\,;\ a, b, c \\ \lambda, \mu, \nu, \rho, \sigma\,;\ \alpha, \beta, \gamma \end{matrix}\right\} = 1, 2, \ldots\ldots\ldots\ldots n,$$
$$H, I, J, K\,;\ A, B, C = 1, 2, \ldots\ldots\ldots\ldots 2n.$$