RIEMANN-CESÀRO METHODS OF SUMMABILITY III

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1. Introduction. Let s_n^α be the Cesàro sum of a series $\sum_{n=0}^{\infty} a_n$ with $a_0 = 0$, that is, A_n^α being Andersen's notation,

$$s_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^\alpha a_\nu,$$

and let σ_n^α be the Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$, that is, $\sigma_n^\alpha = s_n^\alpha / A_n^\alpha$. The series $\sum_{n=0}^{\infty} a_n$ is said to be evaluable (C, α), $\alpha > -1$, to s if $\sigma_n^\alpha \to s$ as $n \to \infty$ and to be evaluable $|C, \alpha|$, $\alpha > -1$, to s if the series $\sum_{n=0}^{\infty} |\sigma_n^\alpha - \sigma_{n+1}^\alpha|$ is convergent and if $\sigma_n^\alpha \to s$ as $n \to \infty$. In the following, let p be a positive integer and let α be a real number such that $\alpha \geqq -1$. The series $\sum_{n=0}^{\infty} a_n$ is said to be evaluable to s by Riemann-Cesàro method of order p and index α, or briefly, to be evaluable (R, p, α) to s, if the series

$$C_{p,\alpha}^{-1} \cdot t^{\alpha+1} \sum_{n=1}^{\infty} s_n^\alpha \left(\frac{\sin nt}{nt} \right)^p, \tag{1.1}$$

where

$$C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty u^{\alpha-p} (\sin u)^p \, du, & -1 < \alpha < p-1 \text{ or } \alpha = 0,\ p = 1, \\ 1, & \alpha = -1, \end{cases} \tag{1.2}$$

converges in some interval $0 < t < t_0$ and its sum tends to s as $t \to 0+$. Under this definition, the summabilities $(R, p, -1)$ and $(R, p, 0)$ are reduced to the well-known summabilities (R, p) and (R_p), respectively. In our ealier papers [3, 4] we have investigated some properties on this summability. The purpose of this paper is to study further properties on this. One of our problems is to establish the inclusion relation between the methods with same order and distinguished indices. Concerning this problem, Marcinkiewicz [6] proved the following theorems.

THEOREM A. *A series may be evaluable (R_2) without being evaluable $(R, 2)$.*