ON THE PROJECTION OF NORM ONE IN W^*-ALGEBRAS, III

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This paper is a continuation of the author's preceding papers [8], [9], in which we discuss certain existence-problems of σ-weakly continuous projections of norm one of different types of W^*-algebras.

By a projection of norm one we mean a projection mapping from a Banach space onto its subspace whose norm is one. In the following we concern with the projection of norm one in a W^*-algebra $\mathbf{M}$. We denote by $\mathbf{M}_*$ the space of all σ-weakly continuous linear functionals on $\mathbf{M}$. On the other hand $\mathbf{M}^*$ means the conjugate space of $\mathbf{M}$ and the second conjugate space of $\mathbf{M}$ is written by $\mathbf{M}^{**}$ usually. However, in case $\mathbf{M}$ is a W^*-algebra $\mathbf{M}^{**}$ is the W^*-algebra that plays a special rôle for $\mathbf{M}$ (cf. [3], [7]) so that we denote especially by $\widetilde{\mathbf{M}}$. A positive linear functional φ on a W^*-algebra is called singular if there exists no non-zero positive σ-weakly continuous functional such as $\psi \leqq \varphi$. The closed subspace of $\mathbf{M}^*$ generated by all singular linear functionals is denoted by $\mathbf{M}_*^{\perp}$. Then we get $\mathbf{M}^* = \mathbf{M}_* \oplus \mathbf{M}_*^{\perp}$: the sum is l^1-direct sum. A uniformly continuous linear mapping π from a W^*-algebra $\mathbf{M}$ to another W^*-algebra $\mathbf{N}$ is called singular if ${}^t\pi(\mathbf{N}_*) \subset \mathbf{M}_*^{\perp}$ where ${}^t\pi$ means the transpose of π.

All other notations and definitions are referred to [7] and [8]. Before going to discussions, the author expresses his hearty thanks to Mr. M. Takesaki for his valuable suggestions and co-operations.

1. General decomposition theorem.

THEOREM 1. *Let $\mathbf{M}$, $\mathbf{N}$ be W^*-algebras, then any uniformly continuous linear mapping from $\mathbf{M}$ to $\mathbf{N}$ is uniquely decomposed into the σ-weakly continuous part and the singular part.*

PROOF. Let π be a uniformly coninuous linear mapping from $\mathbf{M}$ into $\mathbf{N}$, then ${}^t\pi$ is the mapping from $\mathbf{N}^*$ to $\mathbf{M}^*$. Consider the restriction of ${}^t\pi$ to $\mathbf{N}_*$. The transpose of this restriction is a σ-weakly continuous linear mapping $\tilde{\pi}$ from $\widetilde{\mathbf{M}}$ to $\mathbf{N}$ and clearly $\tilde{\pi}$ is a σ-weakly continuous extension of π to $\widetilde{\mathbf{M}}$. Denote by $\mathbf{M}_*^0$ the polar of $\mathbf{M}_*$ in $\widetilde{\mathbf{M}}$, then we get a central projection z in $\widetilde{\mathbf{M}}$ such as $\mathbf{M}_*^0 = \widetilde{\mathbf{M}}(1-z)$.

Put $\pi_1(a) = \tilde{\pi}(az)$, $\pi_2(a) = \tilde{\pi}(a(1-z))$ for each $a \in \mathbf{M}$. We have, clearly, $\pi = \pi_1 + \pi_2$. Moreover we get