A NOTE ON CONTRACTION SEMI-GROUPS OF OPERATORS

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(Received August 31, 1958)

1. Let $\Sigma = \{T(\xi); 0 \leq \xi < \infty\}$ be a one-parameter semi-group of operators from an abstract (L)-space X into itself satisfying the following conditions:

(a) For each $\xi > 0$, $T(\xi)$ is a contraction (transition) operator[1].

(b) $T(\xi + \eta) = T(\xi)T(\eta)$ for each ξ, $\eta \geqq 0$ and $T(0) = I$.

(c) $\lim_{\xi \downarrow 0} T(\xi)x = x$ for each $x \in X$.

Such a semi-group is called a contraction (transition) semi-group of operators. We say that $\Sigma' = \{T'(\xi); 0 \leqq \xi < \infty\}$ dominates $\Sigma = \{T(\xi); 0 \leqq \xi < \infty\}$ if

$$T'(\xi)x \geqq T(\xi)x$$

for each $x \geqq 0$ and $\xi > 0$.

We shall deal with the problem on the generation of contraction semi-groups dominating a given contraction semi-group. This problem has been discussed by G. E. H. Reuter[2].

2. We shall define a linear functional $(e, \cdot)$ by

$$(e, x) = \|x^+\| - \|x^-\| \qquad \text{for each } x \in X. \tag{2. 1}$$

An elementary argument shows that $(e, \cdot)$ is a positive linear functional and $|(e, x)| \leqq \|x\|$ for each $x \in X$.

The following theorem is due to Reuter and is a variant of the Hille-Yosida theorem which will be convenient for our purposes.

THEOREM 1. *A linear operator A with an dense domain $D(A)$ generates a contraction (transition) semi-group if and only if*

(i) $(e, Ax) \leqq 0 \,(= 0)$ *for* $x \geqq 0$ *in* $D(A)$,

(ii) *for each* $\lambda > 0$ *and* $x \in X$, *the equation*

$$\lambda y - Ay = x$$

has a unique solution $y = R(\lambda; A)x \in D(A)$ *and* $R(\lambda; A)x \geqq 0$ *for* $x \geqq 0$.

We shall first prove the following

1) A positive linear operator T on X is called a contraction (transition) operator if $\|Tx\| \leqq \|x\|$ $(\|Tx\| = \|x\|)$ for $x \geqq 0$.

2) A note on contraction semi- groups, Math. Scand., vol. 3, 1955.