ON THE PRODUCT PROJECTION OF NORM ONE IN THE DIRECT PRODUCT OF OPERATOR ALGEBRAS

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In the theory of operator algebras, the following result is known (cf. [2], [4], [9], [14]): If θ_1 and θ_2 are normal $*$-homomorphisms from W^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ onto the other W^*-algebras $\mathbf{N}_1$ and $\mathbf{N}_2$, then there exists the unique normal $*$-homomorphism θ from the W^*-direct product of $\mathbf{M}_1$ and $\mathbf{M}_2$, $\mathbf{M}_1 \otimes \mathbf{M}_2$ onto that of $\mathbf{N}_1$ and $\mathbf{N}_2$, $\mathbf{N}_1 \otimes \mathbf{N}_2$ such as $\theta(x \otimes y) = \theta_1(x) \otimes \theta_2(y)$. Moreover, combining this result with that of Takeda [8], it can be shown that if θ_1 and θ_2 are $*$-homomorphisms from C^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ onto the other C^*-algebras $\mathbf{N}_1$ and $\mathbf{N}_2$ there exists the unique $*$-homomorphism θ from the C^*-direct product of $\mathbf{M}_1$ and $\mathbf{M}_2$, $\mathbf{M}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{M}_2$, onto that of $\mathbf{N}_1$ and $\mathbf{N}_2$, $\mathbf{N}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{N}_2$ such as $\theta(x \otimes y) = \theta_1(x) \otimes \theta_2(y)$. In both cases, if θ_1 and θ_2 are $*$-isomorphisms θ is a $*$-isomorphism, too.

In the present paper we shall show that analogous results also hold for the projections of norm one in the operator algebras. Namely, for two W^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ and their W^*-subalgebras $\mathbf{N}_1$ and $\mathbf{N}_2$ if π_1 and π_2 are normal projections of norm one from $\mathbf{M}_1$ and $\mathbf{M}_2$ to $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively, there exists the unique normal projection of norm one π from $\mathbf{M}_1 \otimes \mathbf{M}_2$ to its W^*-subalgebra $\mathbf{N}_1 \otimes \mathbf{N}_2$ such as $\pi(x \otimes y) = \pi_1(x) \otimes \pi_2(y)$ and similar result holds for the projections of norm one in C^*-algebras. After proving them we shall show some applications of these results in the next section.

Before going into discussions, the author recalls many valuable conversations with Mr. M. Takesaki in the presentation of this paper for which he must thank to him.

1. In the sequel, the algebraic direct product of two operator algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ are always denoted by $\mathbf{M}_1 \odot \mathbf{M}_2$. For two C^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$, the α-norm of an element $\sum_{i=1}^{n} x_i \otimes y_i$ of $\mathbf{M}_1 \odot \mathbf{M}_2$ is defined as follows:

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|^2 =$$