ARITHMETIC OF GROUP REPRESENTATIONS

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Let $\mathfrak{G}$ be a finite group, k be an algebraic field of finite degree over the field of rationals $\mathbf{Q}$. In a representation space V over k we consider a $\Gamma = \mathfrak{o}[\mathfrak{G}]$-lattice (Gitter) M in V which is a regular $\mathfrak{o}$-right module and $\mathfrak{G}$-left module where $\mathfrak{o}$ is the ring of integers in k. The set of all Γ-lattices which we denotes by $\{M; k/\mathfrak{o}\}$ can be classified into Γ-isomorphic Γ-lattices in the following way:

$$\{M;\ k/\mathfrak{o}\} = \{M_1;\ \mathfrak{o}/\mathfrak{o}\} + \cdots\cdots + \{M_c;\ \mathfrak{o}/\mathfrak{o}\}.$$

If $k = \mathbf{Q}$ is the field of rationals and V is irreducible, this class number is always finite and was proved by C. Jordan [13][1].

In the book of Speiser [20] this theorem was proved only in two special cases, namely, $\mathfrak{G}$ is a cyclic group or V is absolutely irreducible. The reason for this may be explained by the following considerations.

Let $\mathfrak{p}$ be a finite or infinite prime. We can consider $\mathfrak{p}$-extension $M_{\mathfrak{p}}$ of the Γ-lattice M and put

$$\{M_{\mathfrak{p}};\ k_{\mathfrak{p}}/\mathfrak{o}_{\mathfrak{p}}\} = \{M_{\mathfrak{p}}^{(1)};\ \mathfrak{o}_{\mathfrak{p}}/\mathfrak{o}_{\mathfrak{p}}\} + \cdots\cdots + M_{\mathfrak{p}}^{(j)};\ \mathfrak{o}_{\mathfrak{p}}/\mathfrak{o}_{\mathfrak{p}}\}.$$

The local class number $j = j(\mathfrak{p})$ is always finite and $= 1$ if $\mathfrak{p}$ does not divide the order $g = \# \mathfrak{G}$ of the group $\mathfrak{G}$.

If we define genus of M as

$$\{M;\ \tilde{\mathfrak{o}}/\mathfrak{o}\} = \bigcap_{\mathfrak{p}} \{M;\ \mathfrak{o}_{\mathfrak{p}}/\mathfrak{o}_{\mathfrak{p}}\}$$

then the number of genera in all Γ-lattices in V is

$$j = \prod_{\mathfrak{p} | g} j(\mathfrak{p})$$

and is finite (§7). If M is absolutely irreducible we have

$$c = j \qquad (\S\ 10).$$

On the other hand, number of classes in a genus is expressible as a kind of class number of a suitable algebraic group (§9), which was considered by T. Ono [17] and its finiteness was proved for commutative case by him. Simple considerations show that if $\mathfrak{G}$ is cyclic and $k = \mathbf{Q}$

1) Number in the bracket refers to the bibliography at the end of this paper.