ON A LEMMA OF MARCINKIEWICZ AND ITS APPLICATIONS TO FOURIER SERIES

SHIGEKI YANO

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1. Introduction. In a series of papers, J. Marcinkiewicz proved several theorems concerning conjugate function [6], strong summability of Fourier series [8], and integrals of Dini type [7]. The crucial point in his proofs is in the applications of a lemma concerning a metrical property of closed sets, which is stated as follows:

Let P be a closed set periodic with period 2π, Δ_n $(n = 1, 2, \ldots\ldots)$ be its contiguous intervals, and $\varphi(x)$ be a function which is equal to $|\Delta_n|$[1] *on Δ_n and vanishes on P, then the integral*

$$\int_{-\pi}^{\pi} \frac{\varphi^{\lambda}(x+t)}{|t|^{\lambda+1}}\,dt, \qquad \lambda > 0,$$

is finite for almost every x in P.

In a paper above cited [7], Marcinkiewicz introduced an integral of Dini type:

$$\mu(f; x) = \left(\int_0^{\pi} \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{t^3}\,dt\right)^{1/2}, \tag{1.1}$$

where $F(x) = \int_0^x f(t)\,dt$, and proved that [2]

$$A\|f\|_2 \leqq \|\mu(f)\|_2 \leqq A\|f\|_2. \tag{1.2}$$

He also conjectured and A. Zygmund [18] proved that

$$A_p\|f\|_p \leqq \|\mu(f)\|_p \leqq A_p\|f\|_p, \qquad 1 < p < \infty, \tag{1.3}$$

where in the left side inequality, we suppose that $\int_0^{2\pi} f(x)dx = 0$. The proof of Zygmund depends on "complex method"; indeed he proved that $\mu(f)$ is majorized by the Littlewood and Paley function g^*[3].

With the aid of the lemma of Marcinkiewicz, Zygmund [17] proved that the Fourier series of an integrable function $f(x)$ is strongly summable almost

1) $|E|$ denotes the measure of a set E.
2) Here and hereafter, A stands for an absolute positive constant, and A_p, etc. stand for constants depending only on the indicated parameters, The constants A and A_p, etc. need not be the sames at different occasions.
3) For the function g^*, see [4] and [18].