ON PRODUCTS OF HOCHSCHILD GROUPS

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(Received December 10, 1958)

Introduction. We shall consider three K-algebras Λ ,Γ, Σ, where K is a commutative ring, and that modules with operators are all unitary. ${}_{\Lambda}A$ means a left Λ-module, and we transfer ${}_{\Lambda}A$ to A_{Λ^*} in many case, where Λ^* is the "opposite" algebra of Λ. In the situation $(A_{\Lambda\otimes\Gamma}, {}_{\Lambda}B_{\Sigma})$[1], we convert $A \otimes {}_{\Lambda}B$ into a right $\Gamma \otimes \Sigma$-module by setting

$$(a \otimes b)(\gamma \otimes \sigma) = a\gamma \otimes b\sigma.$$

Similarly in the situation $({}_{\Lambda}B_{\Sigma}, C_{\Gamma\otimes\Sigma})$ we convert $\mathrm{Hom}_{\Sigma}(B, C)$ into a right $\Lambda \otimes \Gamma$-module by setting

$$(f(\lambda \otimes \gamma))b = (f(\lambda b))\gamma.$$

There are associative formulae:

(1) $r : (A \otimes_{\Lambda} B) \otimes_{\Gamma\otimes\Sigma} C \approx A \otimes_{\Lambda\otimes\Gamma}(B \otimes_{\Sigma} C)$, $\qquad (A_{\Lambda\otimes\Gamma}, {}_{\Lambda}B_{\Sigma}, {}_{\Gamma\otimes\Sigma} C)$
such that $r[(a \otimes b) \otimes c] = a \otimes (b \otimes c)$,

(2) $s : \mathrm{Hom}_{\Lambda\otimes\Gamma}(A, \mathrm{Hom}_{\Sigma}(B, C)) \approx \mathrm{Hom}_{\Gamma\otimes\Sigma}(A \otimes_{\Lambda} B, C)$, $\qquad (A_{\Lambda\otimes\Gamma}, {}_{\Lambda}B_{\Sigma}, C_{\Gamma\otimes\Sigma})$
such that $(sf)(a \otimes b) = (fa)b$ for $f: A \to \mathrm{Hom}_{\Sigma}(B, C)$,

(3) $t : \mathrm{Hom}_{\Lambda\otimes\Gamma}(A, \mathrm{Hom}_{\Sigma}(B, C)) \approx \mathrm{Hom}_{\Sigma\otimes\Gamma}(B \otimes_{\Lambda} A, C)$, $\qquad ({}_{\Lambda\otimes\Gamma}A, {}_{\Sigma}B_{\Lambda}, {}_{\Sigma\otimes\Gamma}C)$
such that $(tg)(b \otimes a) = g(a)b$ for $g : A \to \mathrm{Hom}_{\Sigma}(B, C)$. (p. 165 of [1].)

Since these isomorphisms establish natural equivalences of functors, we get the transition of "projective" or "injective":

PROPOSITION 1 (pp. 165-6 of [1]). *In the situation* $(A_{\Lambda\otimes\Gamma}, {}_{\Lambda}B_{\Sigma})$ *assume that* A *is* $\Lambda \otimes \Gamma$*-projective and* B *is* Σ*-projective. Then* $A \otimes_{\Lambda} B$ *is projective as a right* $\Gamma \otimes \Sigma$*-module.*

PROPOSITION 2 (loc. cit.). *In the situation* $({}_{\Lambda}B_{\Sigma}, C_{\Gamma\otimes\Sigma})$ *assume that* B *is* Λ*-projective and* C *is* $\Gamma \otimes \Sigma$*-injective. Then* $\mathrm{Hom}_{\Sigma}(B, C)$ *is injective as a right* $\Lambda \otimes \Gamma$*-module.*

Prop. 1 and Prop. 2 yield fundamental facts for the product theory, for instance,

PROPOSITION 3. (p. 166 of [1]). *In the situation* $(A_{\Lambda\otimes\Gamma}, {}_{\Lambda}B_{\Sigma})$ *let* X *be a* $\Lambda \otimes \Gamma$*-projective resolution of* A*, and* Y *a* $\Lambda^* \otimes \Sigma$*-projective resolution of* B

1) $\otimes$ means tensor product over K; occasionally the mention of the rings of reference K may be omitted when no confusion can occur.