MEROMORPHIC FUNCTIONS WITH MAXIMUM DEFECT SUM

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1. Introduction. Pfluger [6] proved that if $f(z)$ be an entire function of finite order ρ with maximum defect sum 2, then ρ must be an integer. In this note we extend this theorem to meromorphic functions. We prove

THEOREM. *Let $f(z)$ be a meromorphic function of finite order ρ such that $\delta(a_1) = 1$, $\sum_2^\infty \delta(a_i) = 1$ where a_1, $a_2, \dots$ are any constants (finite or infinite) different from each other. Then ρ must be a positive integer and $f(z)$ must be of regular growth order ρ.*

We show also by means of an example that $f(z)$ need not be of very regular growth order ρ or even proximate order $\rho(r)$.

2. Lemma. *Let $F(z)$ be a meromorphic function of non-integer order $\rho > 0$ and*

$$\limsup_{r\to\infty} \frac{N(r, a) + N(r, b)}{T(r)} = \chi(\rho)$$

where a and b are any two distinct numbers finite or infinite; then if $p = [\rho] > 0$,

$$\chi(\rho) \geqq (\rho - p)(p + 1 - \rho)/\{3e(2 + \log p)(1 + p)^2\}, \tag{1}$$

and if $p = 0$,

$$\chi(\rho) \geqq 1 - \rho \tag{2}$$

Two proofs of this lemma, with different constants,[3] on the right hand sides of (1) and (2), are known [4; pp. 51-54; 10, theorem 2 (a)]. We sketch a different proof depending on the proximate order $\rho(r)$.

Since $T\left(r, \dfrac{\alpha F + \beta}{\gamma F + \delta}\right) = T(r, F) + O(1)$, we may suppose $a = 0$, $b = \infty$. Then

$$F(z) = z^k e^{q(z)} \prod_1^\infty E\left(\frac{z}{a_i}, p_1\right) \Big/ \prod_1^\infty E\left(\frac{z}{b_j}, p_2\right) = z^k e^{q(z)} P_1/P_2 \qquad \text{(say)}$$

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3) The constants on the right hand sides of (1) and (2) are not "best possible."