AN APPROXIMATION PROBLEM PROPOSED BY K. ITÔ

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1. Itô proposed the following problem: *Given a sequence f satisfying*

$$\sum_{n\geqq 0}\frac{\lambda^n}{n\,!}\,|f(n)|^2<+\infty \tag{1. 1}$$

and a number $\varepsilon>0$, does there exist a polynomial P such that

$$\sum_{n\geqq 0}\frac{\lambda^n}{n\,!}\,|f(n)-P(n)|^2\leqq\varepsilon\ ? \tag{1. 2}$$

Izumi has given [1] an affirmative answer by *constructing* such polynomials in the case in which (1. 1) is strengthened to

$$\sum_{n\geqq 0}|f(n)|^2/w^n<+\infty \tag{1. 1'}$$

for some w. This entails the *existence* of the approximating polynomials under the weaker hypothesis (1. 1) because the Dirac sequences satisfy (1. 1′) and are already total amongst those sequences which satisfy (1. 1). Izumi also deals in analogous fashion with the problem in which means with index 1 replace those with index 2 appearing above.

In this paper we give a rapid existential proof, based upon the Hahn-Banach Theorem, of a more general assertion.

2. The set of positive integers is replaced by an arbitrary locally compact space T, summation being replaced by integration with respect to a chosen positive Radon measure μ on T. Assume that $u_1, \ldots\ldots, u_n$ are n real-valued functions on T such that the mapping

$$u:\ t\rightarrow(u_1(t),\ \ldots\ldots,\ u_n(t)) \tag{2. 1}$$

is a homeomorphism of T into R^n (n dimensional real number space), and such that, if $\|u(t)\|=\sum_{k=1}^{n}|u_k(t)|$, then

$$\int_T \exp\,(a\|u(t)\|)\,d\mu(t)<+\infty \tag{2. 2}$$

for a suitable number $a>0$. Finally let p be an exponent satisfying $1\leqq p<+\infty$, and let p' be the conjugate exponent.

THEOREM. *If the mapping* (2. 1) *is a homeomorphism of T into R^n,*