SOME INTEGRABILITY THEOREMS OF TRIGONOMETRIC SERIES AND MONOTONE DECREASING FUNCTIONS

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1. Let $\{\lambda_n\}$ be a decreasing sequence tending to zero as $n \to \infty$, and put

$$g_1(x) = \sum_{n=1}^{\infty} \lambda_n \cos nx, \qquad h_1(x) = \sum_{n=1}^{\infty} \lambda_n \sin nx.$$

Let $g_2(x)$ and $h_2(x)$ be both non-increasing functions bounded below in $(0, \pi)$ and such that

$$xh_2(x) \in L(0, \pi), \qquad g_2(x) \in L(0, \pi). \tag{A}$$

We put $a_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} g_2(x) \cos nx\, dx$, $b_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} h_2(x) \sin nx\, dx$.

Denote by $L(x)$ a slowly increasing function, that is, $L(x)$ is positive, continuous in $x \geqq 0$ and for any fixed $t > 0$,

$$\frac{L(tx)}{L(x)} \to 1 \text{ as } x \to \infty.$$

S. Aljančić, R. Bojanić and M. Tomić established in the paper [2] that $x^{-\gamma}L(1/x)g_1(x) \in L(0, \pi)$ for $0 < \gamma < 1$, if and only if $\sum n^{\gamma-1}L(n)\lambda_n$ converges, and that $x^{-\gamma}L(1/x)h_1(x) \in L(0, \pi)$ for $0 < \gamma < 2$, if and only if $\sum n^{\gamma-1}L(n)\lambda_n$ converges.

D. Adamović proved in the paper [1] that $x^{\gamma-1}L(1/x)h_2(x) \in L(0, \pi)$ for $0 < \gamma < 2$, if and only if $\sum n^{-\gamma}L(n)b_n$ converges absolutely, and that $x^{\gamma-1}L(1/x)$ $g_2(x) \in L(0, \pi)$ for $0 < \gamma < 1$, if and only if $\sum n^{-\gamma}L(n)a_n$ converges absolutely.

And recently Chen Yung-Ming showed the interesting theorems which are related to the above results [3].

In this note, we shall make some inprovement of the inequalities in T. M. Flett [4] and apply it to the generalization of those four theorems.

In this note, the condition (A) is not assumed preliminarily.

If $p = 1$, our theorems $2 - 5$ coincide with the just mentioned theorems.

The method of proof in Theorem 1 is due to T. M. Flett [4]. Theorems $2 - 5$ correspond to Theorems $2 - 5$ in G. Sunouchi [5] and our proofs will go along the line of [5] respectively.