ON THE CLASS OF SATURATION IN THE THEORY OF APPROXIMATION I[1)]

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1. Introduction. In the preceding paper (Sunouchi-Watari [7]), we have determined the class of saturation for various methods of summation in the theory of Fourier series. In this note we shall prove Fourier integral analogues with necessary modification. But we need somewhat complicated arguments because of the non-existence of the Fourier transform for the class $L^p(p > 2)$ and the pointwise indetermination of the transform even if it exists.

Suppose now that a singular integral operator with kernel $k(t)$

$$(1) \qquad T_\xi(t) \equiv T_\xi(t;f) = \frac{1}{\sqrt{2\pi}\,\xi}\int_{-\infty}^{\infty} f(t+u)k\left(\frac{u}{\xi}\right)du$$

exists. In the following line, the norm means (C) or $L^p(p \geqq 1)$ norm. If there are a positive non-decreasing function $\varphi(\xi)$ and a class of functions $\mathfrak{K}$ such that

(1°) $\|T_\xi(t) - f(t)\| = O\{\varphi(\xi)\}$ as $\xi \to 0$ implies that $f(t)$ is an invariant element of the operator $T_\xi(t;f)$,

(2°) $\|T_\xi(t) - f(t)\| = O\{\varphi(\xi)\}$ implies $f(t) \in \mathfrak{K}$,

(3°) For every $f \in \mathfrak{K}$, we have $\|T_\xi(t) - f(t)\| = O\{\varphi(\xi)\}$,

then it is said that the singular integral operator (1) having the kernel $k(t)$ is saturated with the order $\varphi(\xi)$ and the class $\mathfrak{K}$.

Our problem is to determine the order $\varphi(\xi)$ and the class $\mathfrak{K}$ of saturation. Recently P.L. Butzer [2] has solved the saturation problem for some singular integral operators from the general theory of semi-groups. But many popular singular integral operators don't make semi-groups. We shall give here a direct method to determine the class of saturation for the general operators of singular integrals and supply proofs of some conjectures of Butzer.

We suppose that the kernel $k(t)$ has a continuous function $K(u)$ as its Fourier transform. Moreover we suppose that for some positive functions $\varphi(\xi)$ $(\varphi(\xi)\downarrow 0$ as $\xi \to +0)$ and $\psi(u)$, we have

$$(2) \qquad \lim_{\xi\to+0}\frac{1-K(u\xi)}{\varphi(\xi)} = c\psi(u) \qquad (c \neq 0)$$

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