SPECIAL VARIATIONS

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A simple calculus of variations problem involving second derivatives, i. e., the problem of minimizing

$$J=\int_a^b f(x,y,y',y'')\,dx$$

in a class of admissible functions, $y = y(x)$, does not seem to have been treated very fully in the literature [1]. Possibly this is because the problem can be reduced to one involving only first order derivatives by considering the problem of minimizing

$$\int_a^b f(x,y,z,z')\,dx$$

in a class of admissible functions, $y = y(x)$, $z = z(x)$, satisfying the differential equation, $y' - z = 0$. However, the problem treated this way, as a so-called Lagrange problem, is not necessarily equivalent to the original problem [2]. Possibly, too, it is assumed that the usual method of deriving necessary conditions for problems involving only first order derivatives can be used just as easily to obtain similar conditions for higher order problems. The usual method would be to derive the Euler equation

$$f_y - df_{y'}/dx + d^2f_{y''}/dx^2 = 0$$

and then derive the Weierstrass and Legendre conditions by methods depending on the validity of the Euler equation. There is some disadvantage in this as Graves [3] has pointed out He has derived the Weierstrass condition for problems involving first order derivatives without making use of the Euler equation. It would be convenient to have necessary conditions for problems involving second order derivatives derived directly and independently.

By using a special variation we obtain a necessary condition which yields the Weierstrass condition [4]. Another variation gives a generalization of the Legendre condition by assuming only the existence of the generalized partial derivative $f_{y''y''}$. Still another variation gives the usual Euler equation in integrated form [5]. These variations were suggested by polygonal variations used for problems involving first order derivatives [6]. The proofs are