ON SOME STRUCTURES WHICH ARE SIMILAR TO THE QUATERNION STRUCTURE

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In the following we study some properties of differentiable manifolds of class C^∞ which are endowed with three fields of (non trivial) mixed tensors of class C^∞: ϕ_i^h, ψ_i^h and κ_i^h satisfying the following relations:

$$\phi_i^{\ a}\phi_a^{\ h} = \varepsilon_1\delta_i^h,$$
$$\psi_i^{\ a}\psi_a^{\ h} = \varepsilon_2\delta_i^h,$$
$$\kappa_i^{\ a}\kappa_a^{\ h} = \varepsilon_3\delta_i^h,$$
$$\psi_k^{\ a}\phi_a^{\ i} = \varepsilon\,\phi_k^{\ a}\psi_a^{\ i} = -\,\varepsilon_3\kappa_k^{\ i},$$
$$\kappa_k^{\ a}\psi_a^{\ i} = \varepsilon\,\psi_k^{\ a}\kappa_a^{\ i} = -\,\varepsilon_1\phi_k^{\ i},$$
$$\phi_k^{\ a}\kappa_a^{\ i} = \varepsilon\kappa_k^{\ a}\phi_a^{\ i} = -\,\varepsilon_2\psi_k^{\ i},$$

where δ_i^h denotes the Kronecker delta and $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$; $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3$. The above system contains essentially the following four cases:

Case I. $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$; $\varepsilon = -1$. This is the case of the well-known quaternion structure.

Case II. $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = 1$; $\varepsilon = 1$.

Case III. $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$; $\varepsilon = -1$. This case is called by Libermann the quaternion structure of the second kind, and is also called the complex-product structure by T. Nagano.

Case IV. $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$; $\varepsilon = 1$. All of these structures were studied by Ehresmann and Libermann [3]*, the case I was also studied by Obata [7, 8] and Wakakuwa [10], the case III was also studied by Nagano [4].

In § 1, following T. Nagano [4], we define an almost complex structure and two almost product structures on the tangent bundle $T(M)$ of n dimensional affinely connected manifold M and show that if M is itself an almost complex or almost product manifold, then $T(M)$ turns out to have structures mentioned above, under some conditions for the defined almost complex structure to be integrable is also obtained.

In § 2 we study the affine or metric connections which make the given

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*) Numbers in brackets refer to the references at the end of the paper.