SOME REMARKS ON A REPRESENTATION OF A GROUP

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(Received April 2, 1960)

1. In the course of the study of the crossed product of rings of operators, it has been shown in [4] that an arbitrary countable group admits a faithful representation as a group of outer automorphisms[1)] of an approximately finite factor on a separable Hilbert space.

Let G be an arbitrary countably infinite group. Let Δ be the set of all functions $\alpha(g)$ on G : $\alpha(g) = 1$ on a finite subset of G and $= 0$ elsewhere, and Δ is an additive group under the addition $[\alpha + \beta](g) = \alpha(g) + \beta(g)$ (mod 2), $0(g) = 0$ $(g \in G)$. Let Δ' be the set of all functions $\varphi(\gamma)$ on Δ : $\varphi(\gamma) = 1$ on a finite subset of Δ and $= 0$ elsewhere. Δ' is an additive group under the addition $[\varphi + \psi](\gamma) = \varphi(\gamma) + \psi(\gamma)$ (mod 2) and $0(\gamma) = 0$ $(\gamma \in \Delta)$. For every $\alpha \in \Delta$, $\varphi \rightarrow \varphi^{\alpha}$: $\varphi^{\alpha}(\gamma) = \varphi(\gamma + \alpha)$ is an automorphism of Δ'. Defining the product $(\varphi, \alpha)(\psi, \beta) = (\varphi^{\beta} + \psi, \alpha + \beta)$, we have a locally finite countably infinite group $\mathfrak{G}$ of all elements $(\varphi, \alpha) \in (\Delta', \Delta)$ with the identity $(0, 0)$ and $(\varphi, \alpha)^{-1} = (\varphi^{\alpha}, \alpha)$. Let $\mathbf{H}$ be the Hilbert space $l_2(\mathfrak{G})$, and for each $(\varphi, \alpha) \in \mathfrak{G}$ let $V_{(\varphi, \alpha)}$ be the unitary operator on $\mathbf{H}$ defined $[V_{(\varphi,\alpha)} f]((\psi, \beta)) = f((\psi, \beta)(\varphi, \alpha))$. Then the ring of operators $\mathbf{M}$ generated by all $V_{(\varphi,\alpha)}$ is an approximately finite factor. Next, define an operator $T_g(T_g')$ on $\Delta(\Delta')$: $[T_g\alpha]\,(g') = \alpha(gg')$ for all $\alpha \in \Delta$, $g, g' \in G$; $[T_g'\varphi]\,(\gamma) = \varphi(Tg^{-1}\gamma)$ for all $\varphi \in G$, and $g \rightarrow T_g\,(T_g')$ is an anti-isomorphism of G into a group of automorphisms of Δ (Δ'). For each $g \in G$, we define a unitary operator U_g in $\mathbf{H}$ by $[U_g f]((\varphi, \alpha)) = f((T_g'\varphi, T_g\alpha))$, and $g \rightarrow U_g$ is a faithful unitary representation of G and for each $g \in G(\neq e)$ $V_{(\varphi,\alpha)} \rightarrow U_g^{-1} V_{(\varphi,\alpha)} U_g = V_{(T_g'\varphi, T_g\alpha)}$ defines an outer automorphism of $\mathbf{M}$.

The purpose of this paper is to discuss some algebraic properties of this representation.

2. From here to the end of this paper, G is a countably infinite group and $\mathbf{M}$ is an approximately finite factor in $\mathbf{H} = l_2(\mathfrak{G})$ associated with G as in § 1. By the fixed algebra of H, a subgroup of G, we mean the subalgebra of $\mathbf{M}$ composed of all elements of $\mathbf{M}$ which are simultaneously fixed under all members of H (see [1]). In this section we shall determine the fixed algebras of some

1) An automorphism of a ring means a *-automorphism.