ON FULLY COMPLETE SPACES

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In [9], V.Pták discusses open mapping properties of locally convex spaces and shows that the class of B-complete spaces has an essential rôle. Such spaces, which we shall call "*fully complete*" according to [3], seem to share with some kind of mapping properties of Banach spaces. The purpose of the present note is to describe in §1 a few results concerning the range theorems of closed operators in fully complete spaces and in §2 some properties of fully complete spaces. Henceforth, we shall consider locally convex linear topological spaces over the real or complex field and the terminology will refer to [2].

1. Range theorems in locally convex spaces. The following is a consequence of the open mapping theorem ([9]: 4. 7).

THEOREM 1. 1. *Let E be a fully complete space and F a locally convex space. If u is a closed linear operator with domain E_0 in E and range in F and if u is almost open, then $u(E_0)$ is a closed linear subspace of F.*

PROOF. u is open by virtue of the open mapping theorem, and $E/u^{-1}(0)$ is fully complete in the quotient topology. Moreover, since $u^{-1}(0)$ is a subspace of E_0, the quotient topology of E_0 by $u^{-1}(0)$ is identical with the topology induced by $E/u^{-1}(0)$. Now, let v be the induced mapping of u, then $u = v\cdot\varphi_0$ where φ_0 denotes the restriction on E_0 of the canonical mapping of E onto $E/u^{-1}(0)$ and v is one-to-one and open. To prove that v is a closed operator, supposet tha $\{\dot{x}_\alpha \mid \alpha \in A\}$ is a net in $E_0/u^{-1}(0)$ which is convergent to $\dot{x}_0$ in $E/u^{-1}(0)$, and that $v(\dot{x}_\alpha)$ converges to y_0 in F. Then there exists a net $\{x_\alpha \mid \alpha\in A\}$ in E_0 and x_0 in E such that $x_\alpha \in \dot{x}_\alpha$ for all $\alpha \in A, x_0 \in \dot{x}_0$ and $\{x_\alpha\}$ converges to x_0. Therefore we have $v(\dot{x}_\alpha) = u(x_\alpha) \to y_0$, and hence $x_0 \in E_0$ and $y_0 = u(x_0)$, i. e. $x_\alpha \in E_0/u^{-1}(0)$ and $y_0 = v(\dot{x}_0)$.

In the following, we assume that u is one-to-one and $\{y_\alpha \mid \alpha\in A\}$ is a net in $u(E_0)$ such that $y_\alpha \to y_0$ in F. Then $\{x_\alpha \mid \alpha \in A\}$ where $x_\alpha = u^{-1}(y_\alpha)$ is a Cauchy net in E_0, and hence converges to a point x_0 in E. Since u is a closed operator, $x_0 \in E_0$ and $y_0 = u(x_0)$. The proof is completed.

REMARK. Every homomorphic image of a fully complete space is fully