ON CHARACTERISTIC CLASSES DEFINED BY CONNECTIONS

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1. Introduction In his lecture notes [2], Chern proves that the image of the Weil homomorphism of a differentiable principal fibre bundle is the characteristic algebra of the bunble. His proof, based on the theory of invariant integrals of E.Cartan, is rather computational. We shall give here an alternative proof using a theorem of Leray.

To state our result more explicitly, we have to fix our notations. By $(P,P/G,G)$ we denote a (differentiable) principal fibre bundle with total space P, base space P/G and structure group G. For every closed subgroup H of G, the principal bundle $(P, P/H, H)$ is defined in a natural way.

By $I^k(G)$ we denote the space of ad$\cdot G$-invariant homogeneous polynomial functions of degree k defined on the Lie algebra $\mathfrak{g}$ of G. In the same manner as every quadratic form can be identified with a symmetric bilinear form, every element of $I^k(G)$ can be identified with a symmetric multilinear map $\mathfrak{g}\times\dots\times\mathfrak{g}\rightarrow R$ of degree k invariant by ad$\cdot G$ in the following sense:

$$f(\mathrm{ad}\cdot s(X_1),\dots,\mathrm{ad}\cdot s(X_k)) = f(X_1,\dots,X_k),\ f\in I^k(G),\ X_i\in\mathfrak{g},\ s\in G.$$

Set

$$I(G)=\sum_{k=0}^{\infty} I^k(G).$$

Then $I(G)$ is an algebra.

Let ω be a connection form in $(P, P/G, G)$ and Ω its curvature form. For every $f\in I^k(G)$, there exists a (unique) closed differential form $\bar{f}$ of degree $2k$ defined on P/G such that

$$\pi^*(\bar{f}) = f(\Omega,\dots\dots,\Omega),$$

where π is the projection $P\rightarrow P/G$. Denote by $w(f)$ the cohomology class defined by $\bar{f}$. The algebra homomorphism $w{:}I(G)\rightarrow H^*(P/G;\ R)$ is independent of the connection chosen (see [2]) and is called the Weil homomorphism.
Our main result is the following

THEOREM A. *Let G be a compact Lie group (not necessarily connected), T its maximal torus, $N=N_G(T)$ the normalizer of T in G. Together with a*