SURFACES OF GAUSSIAN CURVATURE ZERO IN EUCLIDEAN 3-SPACE.

WILLIAM S. MASSEY

(Received September 2, 1961)

1. Introduction. Books on the classical differential geometry of surfaces in 3-space usually prove a theorem to the effect that a surface of Gaussian curvature 0 is a developable surface or torse. To be more precise, the following two statements are proved: (a) If every point on a surface of Gaussian curvature 0 is a flat point[1] then the surface is a piece of a plane. (b) If no point on a surface of Gaussian curvature 0 is a flat point, then through every point there is a unique asymptotic line, and the tangent plane is constant along this line.[2]

Apparently all such books neglect completely the case of a surface of curvature zero which has both flat and non-flat points. This seems rather strange in view of the fact that many obvious examples illustrate this case. Perhaps most classical differential geometers felt that this case was too complicated and the possibilities were too numerous to obtain interesting results. The following quotation from the footnote to a paper[3] written by the late Professor A. Wintner in 1955 illustrates this attitude:

"Certain difficulties inherent to Euler's definition of a torse are known since Lebesgue's thesis (1902). But it may not be necessary to go such extremes as Lebesgue went (continous but not one-to-one parametrizations) in order to show that the theory of torses is not as simple as it appears from the texts of differential geometry, including the rigor-conscious books. For is it true that if [the Gaussian curvature] $K \equiv 0$ on a surface S of class C^2, then a neighborhood of every point of S can be "ruled" so as to be a torse in Euler's sense also? I can neither prove nor believe this, not even under the assumption that S is of class C^∞, which in view of the possibility of clustering zeros of [the second fundamental form] H (i. e. of flat points where $H^2 = 0 = K$) is hardly stronger than S being of class C^2.…… A counter example, with S of class C^∞, would be the first such instance in the differential geometry of surfaces as to require the full force of (function-theoretical) analyticity —— rather than just C^∞-character."

It is the purpose of this paper to show that interesting and significant results *can* be obtained on surfaces of curvature 0 in the case where there are both flat

1) That is, a point where all the coefficients of the second fundamental form vanish. Such an umbilic point is also called a planar point.

2) For a precise statement and proof of this result under minimum assumptions of differentiability, see theorem (V) of [2].

3) This footnote occurs on p. 355 of [5].