ON THE EXISTENCE OF RIEMANN METRICS ASSOCIATED WITH A 2-FORM OF RANK $2r$

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1. Introduction. In his paper [3], S.Sasaki proved that if a $(2n+1)$-dimensional manifold M^{2n+1} admits a skew symmetric tensor field ϕ_{ij} of rank $2n$ and a vector field ξ^i such that $\phi_{ij}\xi^j = 0$, then we can find a Riemann metric g_{ij} on M^{2n+1} such that the tensor fields $\phi^i_j = g^{ik}\phi_{kj}$, ξ^i, $\eta_j = g_{jk}\xi^k$, and g_{ij} define a (ϕ, ξ, η, g)-structure on M^{2n+1}, i. e., they satisfy the following relations:

$$\begin{aligned}
\text{rank } |\phi^i_j| &= 2n, \\
\xi^i\eta_i &= 1, \\
\phi^i_j\xi^j &= 0, \\
\phi^i_j\eta_i &= 0, \\
\phi^i_j\phi^j_k &= -\delta^i_k + \xi^i\eta_k, \\
g_{ij}\xi^j &= \eta_i, \\
g_{ij}\phi^i_h\phi^j_k &= g_{hk} - \eta_h\eta_k.
\end{aligned}$$

But, from his proof, it seems difficult to know about the differentiability of the metric tensor g_{ij} defined by him, although it is clearly continuous.

In this paper, we shall give another proof of the existence of such metric with the differentiability of the same class as that of ϕ_{ij} as a corollary of a more general theorem to the effect that for a skew symmetric tensor field ϕ_{ij} of rank $2r$, we can find a Riemann metric g_{ij} with the differentiability of the same class as that of ϕ_{ij} such that the non-zero characteristic values of ϕ_{ij} with respect to this metric are only i and $-i$. There is an analogous theorem in the case of almost complex structures, namely, if an even dimensional manifold M^{2n} admits a skew symmetric tensor field ϕ_{ij} of rank $2n$, then we can find a Riemann metric g_{ij} such that tensors $\phi^i_j = g^{ik}\phi_{kj}$ and g_{ij} define an almost Hermitian structure on M^{2n}. And this theorem also follows as a corollary of the above theorem.

In order to prove the theorem we shall use the representation of $GL(n, R)$ as a product space in [1] (cf. [1], p. 14), i. e., the fact that any regular matrix τ may be written in one and only one way as the product $\tau = \sigma\alpha$ of an orthogonal matrix σ and a positive definite symmetric matrix α, and the factors σ, α of the decomposition of τ are continuous functions of τ. But we must use the fact that σ and α are analytic functions of τ, so we first prove this in §2, and in §3 we shall prove our results.