# ON THE DIFFERENTIAL GEOMETRY OF TANGENT BUNDLES OF RIEMANNIAN MANIFOLDS II

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**1. Introduction.** Let $M^n$ be an $n$-dimensional Riemannian manifold and $T(M^n)$ be its tangent bundle. We can introduce to $T(M^n)$ a natural Riemannian metric from the Riemannian metric of $M^n$.[1]

Now, let us denote by $T_1(M^n)$ the set of all unit tangent vectors of $M^n$. As we can reduce the structural group of $T(M^n)$ to $O(n)$, $T_1(M^n)$ may be regarded as a sphere bundle. We shall call it the tangent sphere bundle of $M^n$. As $T_1(M^n)$ is a submanifold of $T(M^n)$, it has a Riemannian metric naturally induced from that of $T(M^n)$. In this paper I shall study on the differential geometry of this $(2n-1)$-dimensional Riemannian manifold $T_1(M^n)$ regarding it as a submanifold of $T(M^n)$, because it is rather simple analytically.

**2. The Riemannian metric and the connection of** $T_1(M^n)$. Let $U$ be a coordinate neighborhood of $M^n$ with coordinates $x^i$ such that $U \times E^n$ is diffeomorphic with $\pi^{-1}(U)$, where $E^n$ is the vector space which is the standard fibre of $T(M^n)$ and $\pi$ is the natural projection of $T(M^n)$ onto $M^n$. If we denote the components of tangent vector of $M^n$ at $x^i \in U$ with respect to the natural frame $\frac{\partial}{\partial x^i}$ by $v^i$, then the ordered set of variables $(x^i, v^i)$ can be regarded as local coordinates of $\pi^{-1}(U)$ which is an open subset of $T(M^n)$.

Suppose the Riemannian metric of $M^n$ is given in $U$ by the quadratic form

$$ds^2 = g_{jk}(x)dx^j dx^k. \tag{2. 1}$$

Then the Riemannian metric of $T(M^n)$ is given in $\pi^{-1}(U)$ by the quadratic form

$$d\sigma^2 = g_{jk}(x)dx^j dx^k + g_{jk}(x)Dv^j Dv^k,\text{[2]} \tag{2. 2}$$

where $Dv^j$ means the covariant differential of $v^j$, i.e.

$$Dv^j = dv^j + \left\{ {j \atop lm} \right\} v^l dx^m. \tag{2. 3}$$

The components of the fundamental metric tensor of $T(M^n)$ in $\pi^{-1}(U)$ can be

1) cf. S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tôhoku Math. J. 10 (1958) pp. 338-354. This paper will be cited as I.

2) Throughout this paper, we use the same notation as in the paper I.