BEST APPROXIMATION BY WALSH POLYNOMIALS

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1. Let Ψ be the system of Walsh functions, in the sense of R. E. A. C. Paley [3]; it is a complete orthonormal system over the unit interval [0,1] and may be considered as the set of the characters of the "dyadic group" of N.J.Fine [1]. Further, G. Morgenthaler [2] introduced the notion of (W)-continuity, i.e. the image in the unit interval of the continuity in the dyadic group, which describes the behavior of the Walsh-Fourier expansion of a function more precisely than the ordinary one does. For example, there is a very simple and complete parallelism between the order of the best approximation by Walsh polynomials and the smoothness (in the sense of (W)-continuity) of a function. However, this parallelism seems to have escaped attention, and since it has interesting applications, we study it in details.

In what follows we shall confine our considerations on the dyadic group; once this is done, the case of the unit interval will then follow without difficulty. Let us list some preliminary definitions and notations; the reader is referred to Fine [1].

The dyadic group G is the set of the sequences $x = (x_n)$, $n = 1, 2, \cdots$, consisting of 0's and 1's, with termwise addition modulo 2, denoted by $\dotplus$. The topology of G is defined by the system of neighborhoods of identity $V_n = \{x \in G\,; x_1 = \cdots = x_n = 0\}$ $(n = 0, 1, 2, \cdots)$ (with a convenience $V_0 = G$), or equivalently by the distance $\lambda(x) = \sum_{n=1}^{\infty} 2^{-n} x_n$. The total measure of G is normalized to be equal to 1.

The Rademacher functions are the "elementary characters" of G:

$$\phi_n(x) = (-1)^{x_{n+1}} \qquad (n = 0, 1, \cdots)$$

and the Walsh functions, the characters of G, are given by

$$\psi_n(x) \equiv 1, \quad \psi_n(x) = \phi_{n_1}(x) \cdots \phi_{n_r}(x) \quad \text{for } n = 2^{n_1} + \cdots + 2^{n_r}$$
$$(n_1 > \cdots > n_r \geqq 0 \quad \text{integers}).$$

A (Walsh) polynomial is a finite linear combination (over complex num-

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