A NOTE ON THE GENERALIZED HOMOLOGY THEORY

FUICHI UCHIDA

(Received November 4, 1963)

G. W. Whitehead [6] has shown that, for any spectrum $\boldsymbol{E}, \widetilde{\mathfrak{H}}(\boldsymbol{E})$ and $\widetilde{\mathfrak{H}}^*(\boldsymbol{E})$ are generalized homology and cohomology theories on the category $\mathfrak{P}_0$ whose objects are finite CW-complexes with base vertex.

In this note, we show that, for any spectrum $\boldsymbol{E}$, $\widetilde{\mathfrak{H}}(\boldsymbol{E})$ and $\widetilde{\mathfrak{H}}^*(\boldsymbol{E})$ are defined on the category $\mathfrak{W}_0$.

1. Let $\mathfrak{W}_0$ be the category of spaces with base point having the homotopy type of a CW-complex, and a map of $\mathfrak{W}_0$ is a continuous, base point preserving map. In this note, we shall use the terms "space" and "map" to refer to objects and maps of $\mathfrak{W}_0$. Let $\mathfrak{W}_0^n$ be the category of n-ads [6].

Let T be the unit interval with base point 0, $\dot{T} = S^0$ be the subspace $\{0, 1\}$ of T, and $S = S^1 = T/\dot{T}$. The *cone* over X is the space $TX = T \wedge X$, and the *suspension* of X is the space $SX = S \wedge X$, where the space $X_1 \wedge \cdots \wedge X_n$ is the n-fold reduced join of the spaces X_i [6].

Let $[X, Y]$ be the set of homotopy classes of maps of X into Y, if $f: X \to Y$, let $[f]$ be the homotopy class of f. Then $[\ ,\ ]$ is a functor on $\mathfrak{W}_0 \times \mathfrak{W}_0$ to the category of sets with base points. If $f: X' \to X$, $g: Y \to Y'$, let

$$f^{\#} = [f, 1]: [X, Y] \longrightarrow [X', Y],$$
$$g_{\#} = [1, g]: [X, Y] \longrightarrow [X, Y'].$$

LEMMA 1.1. *Let X, Y be CW-complexes and $f: X \to Y$ be a continuous one-to-one onto map. Then the map f is a homeomorphism, if and only if, for any open cell τ of Y, there exist finite open cells $\sigma_1, \cdots, \sigma_n$ of X such that $\tau \subset f(\sigma_1 \cup \cdots \cup \sigma_n)$.*

PROOF. If the map f is a homeomorphism, then for any open cell τ of Y, $f^{-1}(\bar{\tau})$ is a compact set in X, and hence $f^{-1}(\bar{\tau})$ is contained in a finite union of open cells $\sigma_1, \cdots, \sigma_n$ of X [4]. Thus τ is contained in $f(\sigma_1 \cup \cdots \cup \sigma_n)$. Conversely, suppose that for any open cell τ of Y, $f^{-1}(\tau)$ is contained in a finite union of open cells $\sigma_1, \cdots, \sigma_n$ of X. Then $\bar{\tau} \subset f(\bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_n)$. Since f is a homeomorphism on a compact set, $f | \bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_n$ is a homeomorphism and hence $f^{-1} | \bar{\tau}$ is continuous. Therefore f^{-1} is continuous.

2. A *spectrum* $\boldsymbol{E}$ is a sequence $\{E_n | n \in Z\}$ of spaces together with a sequence of maps

$$\varepsilon_n: SE_n \longrightarrow E_{n+1},$$