REMARKS ON 4-DIMENSIONAL DIFFERENTIABLE MANIFOLDS

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Let X_4 be 4-dimensional differentiable manifold and let $B(X_4, Y, G)$ be an arbitrary tensor bundle over X_4, where Y is a linear space of dimension 4^{p+q} with coordinates $(y^{i_1 \cdots i_p}_{j_1 \cdots j_q})$[1]. It is well known ([1]) that the structural group G of $B(X_4, Y, G)$ is reducible to the orthogonal group $O(4)$. And if X_4 is orientable, then it is easily seen that G is reducible to $SO(4)$ or one of its subgroups. If especially Y is a 4^2-dimensional linear space with coordinates (y^i_j), then the matrix representation of $SO(4)$ or its subgroup operates on Y as matrix transformations.

The purpose of this note is first to show the existence of two intrinsic (1-1)-type tensor bundles over X_4, which are subbundles of $B(X_4, Y, G)$ and to show the existence or non existence of cross sections of the two intrinsic subbundles wholly depends on the group G (§2). These are owing to the speciality of $SO(4)$.

Secondly, we classify X_4 following the structural group G and study further on each classes case by case (§3 ~ §7).

1. Preliminary. The local subgroups of $SO(4)$ are treated by Ôtsuki [2] in the standpoint of holonomy groups of 4-dimensional Riemannian manifolds. And the classification of structural equations of all connected subgroups of $SO(4)$ is done by Ishihara [3] making use of the structural equation of $SO(4)$ indicated by Chern [4]. We will consider it in another point of view and will do the classification of the connected subgroups of $SO(4)$ in a different way.

As is known, $SO(4)$ is locally represented as $SO(4)=SU(2)\otimes SU(2)$. $SU(2)$ leaves invariant an anti-involution of the second kind and $SU(2)\otimes SU(2)$ leaves invariant that of the first kind which is the Kronecker product of the anti-involutions left invariant by the two $SU(2)$ (Cartan [5] ; Berger [6]). $SO(4)$ is the real representation of the group $SU(2)\otimes SU(2)$ restricted on the double element (real dimension 4) of the anti-involutions (see Appendix 1°). Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the complexifications of the Lie algebras of the first and the second $SU(2)$. $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are of complex dimension 3. Then $\mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_2$ (direct sum)

1) Throughout this paper, the indices i_1, j_1, $i, j, a, b, \cdots$ run from 1 to 4, unless otherwise stated. This tensor is of type $(p-q)$.