ALMOST-CONVERGENT DOUBLE SEQUENCES

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1. Introduction. In this note we extend to double sequences certain results obtained by Lorentz [4] for simple sequences. We begin with the following summary of the Lorentz results. We recall the existence of Banach limits L_1 [1, p. 34] defined for each element in the Banach space $(m)_1$ of all bounded real sequences $f = \{f(k)\}$ with $\| f \| = \sup_k | f(k)|$. These limits have the following familiar properties.

$$L_1(af + bg) = aL_1(f) + bL_1(g), \quad (\textit{all real } a, b); \tag{1. 1}$$

$$L_1(f) \geqq 0 \quad \textit{if all } f(k) \geqq 0; \tag{1. 2}$$

$$L_1(f_1) = L_1(f) \quad \textit{where } f_1 = \{f(k+1)\}; \tag{1. 3}$$

$$L(e) = 1 \quad \textit{where } e(k) = 1 \textit{ for all } k. \tag{1. 4}$$

If we introduce the positively homogeneous and subadditive functional,

$$q(f) = \inf_{n_1, n_2, \ldots n_p} \limsup_{k} \frac{1}{p} \sum_{i=1}^{p} f(n_i + k), \tag{1. 5}$$

for $f \in (m)_1$, and set $q'(f) = -q(-f)$, then the inequality $q'(f) \leqq L_1(f) \leqq q(f)$ holds for all $f \in (m)_1$ and all Banach limits L_1. Furthermore, all Banach limits will coincide at f if and only if $q'(f) = q(f)$. Sequences f satisfying this condition are called *almost-convergent*, and we denote the class of all such sequences by $(ac)_1$. In order that $q'(f) = q(f)$ it is necessary and sufficient that the *sliding* $(C, 1)$-means of f,

$$\frac{1}{p} \sum_{i=1}^{p} f(n + i), \tag{1. 6}$$

converge uniformly in n as $p \to \infty$. If this condition is satisfied the limit over p in (1. 6) is equal to $L_1(f)$ for every Banach limit L_1. Finally, any sequence whatever such that the means (1. 6) converge uniformly is necessarily bounded.

The problem of extending these results to double sequences $f = \{f(i, j)\}$