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# THE DE RHAM THEOREM FOR GENERAL SPACES*

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It is known that the differential forms on a differentiable manifold $X$ may be defined as a species of singular real-valued cochains[1] on $X$. Now let $X$ be an arbitrary topological space and $\mathfrak{F}$ a set of continuous real-valued functions on $X$. As will be seen in the sequel, one can again single out a species of real singular cochains on $X$ by letting $\mathfrak{F}$ play the role of a differentiable structure[2], and obtain thus a graded differential exterior algebra $G$ associated with the pair $(X, \mathfrak{F})$. Moreover, such pairs can be regarded as objects of a local category[3] $\mathfrak{D}$, in which case $G$ becomes a contravariant functor on $\mathfrak{D}$ with values in the category $\mathfrak{A}$ of graded differential algebras. By a sheaf-theoretic process, $G$ generates a functor $F$ from $\mathfrak{D}$ to $\mathfrak{A}$ of the kind previously referred to as a sheaf[4] on $\mathfrak{D}$. This sheaf $F$ constitutes an extension of the classical differential forms (regarded as a functor on the local category of differentiable manifolds). In the present paper we shall be concerned with the question under what conditions the cohomology of the complex $F(X)$ reduces to the real sheaf cohomology[5] of the underlying space $X$. It will be seen that this holds for objects $X$ lying in a certain subcategory $\mathfrak{C}$ of $\mathfrak{D}$, which however is considerably larger than the category of differentiable manifolds. One has obtained in this way a generalized version of the de Rham Theorem.

Nonclassical objects in $\mathfrak{D}$ arise in various ways, e.g., as quotients of a differentiable manifold $M$. More precisely, every quotient space $X$ of $M$ carries a natural differentiable structure $\mathfrak{F}$ (in the sense referred to above).

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1) From this point of view, the theory of differential forms was extended to Lipschitzian manifolds by Whitney (see [7]).

2) Strictly speaking, we shall find it convenient to deal only with sets $\mathfrak{F}$ satisfying an appropriate closure condition.

3) For basic definitions regarding local categories we refer to Eilenberg [2].

4) See Clifton and Smith [1], p. 446.

5) This cohomology is defined in terms of the *canonical resolution* of the *simple sheaf with fibre* $R$ (the group of real numbers). See Godement [3], p. 173. We shall not be concerned with general *families of support* $\Phi$, but will always suppose $\Phi$ to be the family containing $X$ itself.