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SOME REMARKS ON ANDÔ'S THEOREMS

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1. In [1] T. Andô has proved the following result.

THEOREM A. *Let T be a compact operator on a Hilbert space H. Then every subspace which is invariant under T reduces T if and only if T is a normal operator.*

The purpose of this paper is to remark that the above theorem is generalised to an operator such as T^m is compact for some integer $m \geqq 0$ and to prove a related result.

2. In the sequel, an operator means a bounded linear operator on a Hilbert space H. We denote by $\sigma(T)$ the spectrum and by $\sigma_p(T)$ the point spectrum of an operator T. $\mathfrak{N}_T(\lambda)$ means the λ-th proper subspace of an operator T and $P_{\mathfrak{M}}$ is the orthogonal projection onto a closed subspace $\mathfrak{M} \subset H$.

The following lemma is essentially proved in [5], but we give a proof for convenience' sake.

LEMMA 1. *Let T be an operator such as T^m is compact for some integer $m \geqq 0$. Then $\mu \in \sigma(T) \cap \{\lambda : |\lambda| = \|T\|\}$ implies $\mu \in \sigma_p(T)$.*

PROOF. If $\mu \in \sigma(T)$ and $|\mu| = \|T\|$, there exists a sequence $\{x_n\}$ of unit vectors in H such as $\|Tx_n - \mu x_n\| \to 0$ $(n \to \infty)$. Since T^m is a compact operator, we may assume that (if necessary, by choosing a suitable sub-sequence) the sequence $\{T^m x_n\}$ converges to a certain vector $x \in H$. Then we have

$$\|T^{m-1}x_n - \frac{1}{\mu}T^m x_n\| \leqq \frac{\|T^{m-1}\|}{|\mu|}\|Tx_n - \mu x_n\| \to 0 \quad (n \to \infty)$$

and

$$\|T^{m-1}x_n - \frac{1}{\mu}x\| \leqq \|T^{m-1}x_n - \frac{1}{\mu}T^m x_n\| + \frac{1}{|\mu|}\|T^m x_n - x\| \to 0 \quad (n \to \infty).$$