Tôhoku Math. Journ.
Vol. 19, No. 3, 1967

DUALITY OF CYCLIC MODULES

TOYONORI KATO

(Received June 12, 1967)

Quasi-Frobenius rings which possess so many interesting properties, have been studied by a number of authors. J. Dieudonné pointed out that the duality of R-modules is most closely related to a quasi-Frobenius ring R, stating that, for a both right and left Noetherian ring R, R is quasi-Frobenius if and only if all left and all right finitely generated modules over R are reflexive [3] (see also [5]). H. Bass introduced the terminologies "reflexive" and "torsionless" [4]. These seem most important in duality theory. In this paper we shall study the duality of cyclic modules over rings without any finiteness assumptions generalizing the above theorem (Theorem 12 and Theorem 15) and the Ikeda-Nakayama's theorem [1] (Theorem 13, Corollary 14, and Theorem 15). It seems to me that the duality of cyclic modules is essential in duality theory.

1. Introduction. Throughout this paper, we shall assume that R is a ring with identity element and that every module over R is unitary. If A is a left (right) R-module, the dual $A^* = \mathrm{Hom}_R(A, R)$ becomes a right (left) R-module ([6], p. 65). Thus the dual operation $*$ is a contravariant left exact functor on the category of R-modules to that of R-modules. Considering the element of A as homomorphisms from A^* to R, we get the natural R-homomorphism

$$\delta_A\colon A \longrightarrow A^{**}.$$

We shall say that A is torsionless if δ_A is a monomorphism and reflexive if δ_A is an isomorphism. A is torsionless if and only if $A \subset \Pi R$ (direct product of copies of R) ([6], p. 68). It is well known that every finitely generated projective module P is reflexive and that P^* is also finitely generated projective ([6], p. 68). If we are given the diagram $A \to B$, then we have the following commutative diagram

$$\begin{array}{ccc} A^{**} & \longrightarrow & B^{**} \\ \delta_A \uparrow & & \uparrow \delta_B \\ A & \longrightarrow & B\,. \end{array}$$