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REMARKS ON GROTHENDIECK RINGS

KÔJI UCHIDA

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R.G.Swan has obtained several important results on Grothendieck rings of a finite group. In this note we derive generalizations of some of his results. Throughout this note, R denotes a noetherian integral domain and K denotes its quotient field. All modules we consider are finitely generated unitary left modules. If A is a finite R-algebra (or K-algebra), $G(A)$ denotes the Grothendieck group of A-modules, $P(A)$ denotes the Grothendieck group of projective A-modules, and $C_0(A)$ its reduced class group, i.e, the subgroup of $P(A)$ generated by the elements of the form $[P]-[Q]$, where P, Q are projective and $K\otimes_R P\cong K\otimes_R Q$.

1. R is called regular if its localization $R_\mathfrak{p}$ is a regular local ring for each prime ideal $\mathfrak{p}$. A regular domain is integrally closed [1. Proposition 4.2]. In this section we calculate $G(R\pi)$ for a regular domain R of prime characteristic p and for any finite group π.

PROPOSITION 1. *Any finitely generated module over a regular domain R has a finite projective dimension.*

PROOF. Let M be such a module and let

$$\rightarrow X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow M \rightarrow 0$$

be its projective resolution, where we assume every X_k is finitely generated. Let Y_n be the kernel of d_n. Then Y_n is a finitely generated torsion-free R-module. To show that some Y_n is projective, we first prove the following lemma.

LEMMA. *Let R be an integral domain (not necessarily noetherian), and Y be a finitely generated torsion-free R-module. Let $\mathfrak{p}$ be a prime ideal of R. If $Y_\mathfrak{p}=R_\mathfrak{p}\otimes_R Y$ is $R_\mathfrak{p}$-projective, then $Y_\mathfrak{q}$ is $R_\mathfrak{q}$-projective for each $\mathfrak{q}$ which does not contain a certain element $r\notin\mathfrak{p}$.*

PROOF. Let $F\xrightarrow{f}Y\rightarrow 0$ be exact where F is a finitely generated free R-module. Then the sequence $F_\mathfrak{p}\xrightarrow{f_\mathfrak{p}}Y_\mathfrak{p}\rightarrow 0$ splits by assumption, and we have a