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NON-COMMUTATIVE EXTENSION OF LUSIN'S THEOREM

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1. Introduction. Many important theorems in measure theory have been extended to operator algebras by many authors, especially, Dixmier [1], Dye and Segal. Considered as non-commutative extensions, those are interesting themselves and provide powerful tools in the further investigations of operator algebras. The purpose of this paper is to extend Lusin's theorem which is an important tool in measure theory into general operator algebra.

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2. Notations and Definitions. Let M be a W^*-algebra, namely, C^*-algebra with a dual structure as a Banach space, M_* be the predual of M, that is, the Banach space of all bounded normal functionals on M, and $M_*{}^+$, the positive part of M_*, that is, the set of all functionals φ in M_* such that $\varphi(x^*x) \geqq 0$ for all $x \in M$. We may consider the s^*-topology, that is, the topology defined by a family of semi-norms $\{\alpha_\varphi, \alpha_\varphi^*; \varphi \in M_*{}^+$, where $\alpha_\varphi(x) = \varphi(x^*x)^{1/2}$, and $\alpha_\varphi^*(x) = \varphi(xx^*)^{1/2}$ for all $x \in M\}$, and the s-topology is that defined by a family of semi-norms $\{\alpha_\varphi; \varphi \in M_*{}^+\}$. In [4, p. 1.64] Sakai shows that whenever M is represented as a weakly closed algebra of operators on some Hilbert space, the weak*-topology of M coincides with the weak operator topology on the bounded sets of M. It follows from this that the s^*- topology coincides with the strong $*$-operator topology on bounded sets of M, and the s-topology coincides with the strong operator topology on bounded sets of M.

3. Main theorems. The following theorem corresponds to the Egoroff theorem in the Lebesgue integration.

THEOREM 1. (Density theorem) *Let M be a W^*-algebra and M_* the predual of M, moreover, let φ be any positive functional in M_*. Let N be any set in S (the unit sphere of M), which is adherent to an element a in S in the s^*-topology. Then for any positive number ε, and a projection e in M, there exist a projection e_0 in M and a sequence $\{a_i\}_{i=1}^\infty \subset N$ such that $e \geqq e_0, \varphi(e-e_0) < \varepsilon$ and $\lim_{i\to\infty} \|a_ie_0 - ae_0\| = 0$. In particular, for any sequence*