Tôhoku Math, Journ.
Vol. 19, No. 3, 1967

ON THE PREDUALS OF W^*-ALGEBRAS

KAZUYUKI SAITÔ

(Received May 22, 1967)

In the present paper ,we shall show some properties of weakly relatively compact subsets of predual of W^*-algebra, which were also discussed in [1], [10] and [12].

Let M be a W^*-algebra (namely, C^*-algebra with a dual structure as a Banach space [7]), M^* (resp. M_*) be the dual (resp. predual) of M, and let M_h, M_p, and M_{pi} be the set of all Hermitian elements, projections, and partial isometries in M, respectively.

The weak topology on M_* is $\sigma(M_*, M)$-topology in the sense of [3; p. 50].

For any linear functional φ in M, we define the functionals φa, $a\varphi$, φ^* and $|\varphi|$ on M as follows: $\varphi a(b) = \varphi(ab)$,$a\ \varphi(b) = \varphi(ba)$, $\varphi^*(b) = \overline{\varphi(b^*)}$ for all b $\in M$, where $\overline{\varphi(b^*)}$ is the complex conjugate of $\varphi(b^*)$. $|\varphi|$ is said the absolute value of φ [8]. If φ is in M_*, then φa, $a\varphi$, and φ^* are also in M_*. We denote the set $\{|\varphi|; \varphi \in K\}$ by $|K|$.

A functonal φ on M is positive if $\varphi(a^*a) \geqq 0$ for all $a \in M$. Denote the set of all positive functionals in M^* (resp. M_*) by M^{*+} (resp. M_*^+).

We may consider the following five typical topologies on M:

(1) The norm topology as a Banach space, (2) The Mackey topology τ on M, namely, the togology of uniform convergence on the weakly relatively compact sets of M_*, (3) The topology s^* defined by a family of semi-norms $\{\alpha_\varphi, \alpha_\varphi^*; \varphi \in M_*{}^+\}$, where $\alpha_\varphi(x) = \varphi(x^*x)^{1/2}$, and $\alpha_\varphi^*(x) = \varphi(xx^*)^{1/2}$ for $x \in M$, (4) The topolcgy s defined by a family of semi-norms $\{\alpha_\varphi; \varphi \in M_*{}^+\}$, (5) The weak topology on M as point, which is merely called σ-topology. The topology s^*(resp. s and σ) coincides with strong $*$-operator topology, namely the operator topology defined by a family of semi-norms $\{\|x\xi\|\ \|x^*\xi\|\ ; \xi \in \mathfrak{H}\}$ (resp. the strong operator topology and the weak operator topology) on bounded spheres, when M is faithfully represented as a von Neumann algebra on a Hilbert apace $\mathfrak{H}$. The τ-topology is equivalent to the s^*-topology on bounded spheres. [1]

In the followings, theorem 1 shows a characterization of the finiteness of W^*-algebras. Theorem 2 and the following remark concern with a weak convergence property in the predual of an atomic W^*-algebra, which is a non-commutative generalization of a well known theorem in the Lebesgue L^1, and the last theorem 3 deals with weakly relatively compact subsets lying in