Tôhoku Math. Journ.
Vol. 19, No. 3, 1967

WEIGHTED AVERAGES OF SUBMARTINGALES

NORIHIKO KAZAMAKI AND TAMOTSU TSUCHIKURA

(Received April 20, 1967)

Let $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ be a submartingale. For a given sequence of positive numbers $w_1, w_2, \cdots$, we consider the weighted averages

$$s_n = (w_1x_1 + \cdots + w_nx_n)/W_n \qquad (n = 1, 2, \cdots)$$

where $W_n = w_1 + \cdots + w_n$. Although the sequence $\{s_n\}$ need not be a submartingale, we may expect some similar properties to the original submartingale.

THEOREM 1. *For the submartingale $\{x_n, \mathfrak{F}_n, n \geqq 1\}$, using the above notations, suppose that $\lim_{n\to\infty} W_n = \infty$. Then the following two conditions are equivalent to each other*:

$$(1) \qquad \sup_n \mathrm{E}\{|s_n|\} < \infty\,,$$

$$(2) \qquad \sup_n \mathrm{E}\{|x_n|\} < \infty\,.$$

By the classical submartingale convergence theorem, the condition (2) *is sufficient to insure the almost sure convergence of $\{x_n\}$, hence so is the condition* (1).

PROOF. It is easy to get (1) from (2), in fact,

$$\mathrm{E}\{|s_n|\} \leqq \mathrm{E}\left\{\frac{w_1|x_1| + \cdots + w_n|x_n|}{W_n}\right\} \leqq \sup_j \mathrm{E}\{|x_j|\}\,.$$

To show that (1) implies (2) we consider the two cases of martingale and submartingale.

(i) Let $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ be a martingale. If $m < n$, we have by the definition of conditional expectations and the martingale equality

$$\begin{aligned}\mathrm{E}\{|s_n|\} &= \mathrm{E}\{\mathrm{E}\{|s_n|\,|\mathfrak{F}_m\}\}\\ &\geqq \mathrm{E}\{|\mathrm{E}\{s_n|\mathfrak{F}_m\}|\}\end{aligned}$$