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NEARLY NORMAL OPERATORS

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1. We say that a bounded linear operator T on a Hilbert space H is nearly normal if $T \leftrightarrow T^*T$ where the symbol $\leftrightarrow$ denotes commutativity. $R(T)$ denotes the smallest von Neumann algebra containing T and $R(T)'$ its commutant. The terminology of von Neumann algebras will be found to conform with [2]. In [4], N. Suzuki proved that $R(V)$ is of type I if V is an isometry.

The purpose of this note is to prove that $R(T)$ is also of type I if T is nearly normal. Clearly, isometries are nearly normal.

2. For our object, the following Lemma 1 and Lemma 4 are essential.

LEMMA 1. *If T is a nearly normal operator on H and if E is the projection from H on $\mathfrak{N}_T = \{x \in H\ ;\ Tx=0\}$, then $E \in R(T) \cap R(T)'$.*

PROOF. Clearly, $E \in R(T)$ and $\mathfrak{N}_T$ is invariant under T. Hence we have only to prove $\mathfrak{N}_T$ is invariant under T^*. For any $x \in \mathfrak{N}_T$, we have $\|TT^*x\|^2 = (TT^*x, TT^*x) = ((T^*T)T^*x, T^*x) = (T^*(T^*T)x, T^*x) = 0$ by the definition of nearly normal operators; hence $T^*x \in \mathfrak{N}_T$.

The following lemma is a modification of the results of A. Brown [1].

LEMMA 2. *If T is a nearly normal operator on H such that $\mathfrak{N}_T = (0)$, then, in the polar decomposition $T = V(T^*T)^{1/2}$ of T, V is an isometry and $V \leftrightarrow (T^*T)^{1/2}$.*

PROOF. $\mathfrak{N}_T = (0)$ means that the closure of $(T^*T)^{1/2}H$ is equal to H and this implies V is an isometry. On the other hand $T \leftrightarrow T^*T$ implies $\{(T^*T)^{1/2}V - V(T^*T)^{1/2}\}(T^*T)^{1/2} = 0$. Hence $V \leftrightarrow (T^*T)^{1/2}$.

If V is an isometry, then we have easily $V^m\mathfrak{N}_{V^*} \perp V^n\mathfrak{N}_{V^*}$ for all non-negative integers m, n, $m \neq n$; hence we have the following lemma.

LEMMA 3. *If V is an isometry on H and if E is the projection from H on $\bigoplus_{n=0}^{\infty} V^n\mathfrak{N}_{V^*}$, then $E \in R(V) \cap R(V)'$ and $V|(I-E)H$ is unitary, where $V|(I-E)H$ denotes the restriction of V on its reducing subspace $(I-E)H$*