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ALMOST AUTOMORPHIC AND ALMOST PERIODIC SOLUTIONS WHICH MINIMIZE FUNCTIONALS

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1. Introduction. In this note we give a sufficient condition for the existence of an almost periodic solution of a system of equations. Most of the sufficient conditions in the literature are either in terms of stability or are uniqueness theorems that use a result of Amerio [1]. Our condition is that certain solutions minimize a functional. An early example of such a condition goes back to Favard. [2]. In the proof of Theorem 2 he shows that a certain linear nonhomogeneous system has a unique solution with minimum norm. This solution turns out to be almost periodic. We show how to systematize this argument to get generalizations of some of Favard's results.

2. Definitions and preliminary results. If $\{\alpha'_n\}$ is a sequence we write it as α'. If $\alpha = \{\alpha_n\}$ is a subsequence of α' we write $\alpha \subset \alpha'$. For vector functions of a real variable, the symbol $T_\alpha f(t) = \lim_{n\to\infty} f(t+\alpha_n)$ and it is used only if the indicated limit exists. The sense in which the limit exists will always be indicated. Bochner [3] was the first to notice that almost periodic functions (=a.p.) are precisely those functions f for which given α', β' sequences, there exist $\alpha \subset \alpha'$, $\beta \subset \beta'$ such that $T_{\alpha+\beta} f = T_\alpha(T_\beta f) = T_\alpha T_\beta f$ pointwise, where $\alpha+\beta = \{\alpha_n+\beta_n\}$. It is this characterization that we use in this note.

In the same paper, Bochner introduces the notation of an almost automorphic (=a.a.) function. A bounded function f is a.a. if for every sequence α' there exists $\alpha \subset \alpha'$ such that $T_\alpha f = g$ and $T_{-\alpha} g = f$ exist pointwise. Here $-\alpha = \{-\alpha_n\}$.

We introduce a slightly stronger concept. A bounded function f is compact almost automorphic (compact a.a.) if the above limits are required to exist uniformly on every compact subset of the reals.