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ON THE TYPE OF AN ASSOCIATIVE H-SPACE OF RANK TWO

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An associative H-space is a space X equipped with a continuous map $\mu: X \times X \to X$ providing X with the structure of a monoid. If X is an associative H-space and $H_*(X; \boldsymbol{Z})$ is finitely generated as an abelian group, then by a classical theorem of Hopf [4], [5], $H^*(X, \boldsymbol{Q})$ is an exterior algebra on a finite number of odd dimensional generators. The number of such generators is called the rank of X; this is consistent with the standard usage when X is a Lie group. The dimensions in which the generators occur is called the type of X.

For example $SU(3)$ has rank 2 and type $(3, 5)$.

THEOREM. *Let X be a connected associative H-space with $H_*(X, \boldsymbol{Z})$ finitely generated as an abelian group. If the rank of X is* 2 *then the type of X is either* $(1,1)$, $(1,3)$, $(3,3)$, $(3,5)$, $(3,7)$ *or* $(3,11)$.

Indeed, each of the above types does occur, examples being given by the compact Lie groups, $S^1 \times S^1$, $S^1 \times S^3$, $S^3 \times S^3$, $SU(3)$, $Sp(2)$ and G_2 respectively.

The proof of the above theorem will be accomplished by applying a result of A. Clark [1] and some number theoretic considerations.

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1. Unstable Polyalgebras over $\mathcal{A}^*(p)$.

NOTATION. Let p be a prime. We denote by $\mathcal{A}^*(p)$ the mod-p Steenrod algebra [8]. The reduced p^{th}-powers are denoted by P_p^j, and the Bockstein by β. When $p=2$ we set $\beta = Sq^1$ and $P_2^j = Sq^{2j}$.

DEFINITION. An unstable algebra over the Steenrod algebra is an algebra B that is a left $\mathcal{A}^*(p)$-module satisfying

(1) $$P_p^n x = 0 \quad \text{if} \quad 2n > \deg x$$

(2) $$P_p^n x = x^p \quad \text{if} \quad 2n = \deg x\,,$$

(3) $$P_p^n(xy) = \Sigma_{i+j=n} P_p^i x \, P_p^j y \quad (P_p^0 = 1)\,,$$