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UNIQUENESS OF THE COMPLETE NORM TOPOLOGY AND CONTINUITY OF DERIVATIONS ON BANACH ALGEBRAS*

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1. We are concerned with the following two propositions regarding a complex Banach algebra A:

$\mathcal{N}$ A has unique complete norm topology

$\mathcal{D}$ Derivations on A are necessarily continuous.

Note that if A does not satisfy $\mathcal{N}$ the validity of $\mathcal{D}$ may depend on the choice of topology, cf. the example given below.

It has been known for some time that C^*-algebras satisfy $\mathcal{N}$ [10, Corollary 4.1.18] and $\mathcal{D}$ [11], and also that commutative semi-simple Banach algebras satisfy $\mathcal{N}$ [10, Corollary 2.5.18]. Recently Johnson [6, Theorem 2] has shown that these latter satisfy $\mathcal{D}$ also, so that by [12, Theorem 1] the zero map is the only derivation on such algebras. Subsuming these results Johnson [5] proved that semi-simple Banach algebras satisfy $\mathcal{N}$, and, in the joint paper [7], that these algebras also satisfy $\mathcal{D}$.

Clearly $\mathcal{N}$ and $\mathcal{D}$ are not true in general, since any Banach space can be made into a Banach algebra by defining all products to be zero, and in this situation any norm is an algebra norm, and any linear operator is a derivation. A less extreme example is the following. Let A be the algebra of [4, p. 771] so that $A = l^2 \oplus R$ algebraically, under a norm $\|\cdot\|$ such that l^2 is dense in A, and R is the principal ideal generated by an element r such that $r^2 = rx = 0$, $x \in l^2$. The function $x+\alpha r \mapsto \|x\|' + |\alpha|$, where $\|\cdot\|'$ is the usual l^2 norm is easily seen to be a Banach algebra norm on A, inequivalent to $\|\cdot\|$ since l^2 is $\|\cdot\|'$-closed. Thus A does not satisfy $\mathcal{N}$. Now let D be a derivation on A, so that $D(x)y + xD(y) = D(xy) = D((x+\alpha r)y) = D(x)y + \alpha D(r)y + xD(y)$ for all $x, y \in l^2$, $\alpha \in C$, and so $D(r) = \mu r$ for some constant μ. Thus $D(R) \subseteq R$ and so D induces a derivation $\overline{D}$ on the semisimple algebra A/R. It follows that $\overline{D}$ is zero and hence that $D(A) \subseteq R$. Thus if $x, y \in l^2$, $D(xy) = D(x)y + xD(y) = 0$ so

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