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ON THE EXISTENCE OF NON-COMPARABLE HOMOGENEOUS TOPOLOGIES WITH THE SAME CLASS OF HOMEOMORPHISMS

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Let $H(X, \mathcal{U})$ be the class of all homeomorphisms from $(X, \mathcal{U})$ onto itself. We have already known that, in general, there exist many topologies $\mathcal{V}$ on X such that $H(X, \mathcal{U}) = H(X, \mathcal{V})$[1][2][3]. If $(X, \mathcal{U})$ is an n-manifold, then $H(X, \mathcal{U})$ can also determine, to a certain degree, the topologies $\mathcal{V}$ on X with $H(X, \mathcal{U}) = H(X, \mathcal{V})$ [4]. However it was unknown that whether there are topologies $\mathcal{U}$ and $\mathcal{V}$ on a set X such that $H(X, \mathcal{U}) = H(X, \mathcal{V})$, $(X, \mathcal{U})$ and $(X, \mathcal{V})$ are homogeneous spaces and $\mathcal{U}$ and $\mathcal{V}$ are non-comparable in the sense that there does not exist a permutation Φ on X such that $\{\Phi(U) | U \in \mathcal{U}\} \subseteqq \mathcal{V}$ or $\{\Phi(V) | V \in \mathcal{V}\} \subseteqq \mathcal{U}$. A negative answer would characterize a homogeneous space by means of the class of homeomorphism. The finding of this paper proves the existence of non-comparable homogeneous topological spaces with the same class of homeomorphisms. However it is still open whether there exist non-comparable compact homogeneous topologies with the same class of homeomorphisms.

THEOREM 1. *Let $(X, \mathcal{U})$ be an n-manifold and let $\mathcal{V} = \{U \in \mathcal{U} | X \backslash U$ is compact$\}$ and let $\mathcal{W}$ be the topology having*

$$\left\{ V \backslash \bigcup_{i=1}^{\infty} \{p_i\} \,\middle|\, V \in \mathcal{V} \text{ and } \{p_i\} \textit{ converges to } p_0 \textit{ in } (X, \mathcal{U}) \right\}$$

as a subbase. Then $H(X, \mathcal{U}) = H(X, \mathcal{W})$. If $(X, \mathcal{U})$ is not compact, then $(X, \mathcal{U})$ and $(X, \mathcal{W})$ are non-comparable.

Theorem 1 is a special case of a more general theorem. Let $(X, \mathcal{U})$ be a topological space, a topology $\mathcal{V}$ on X is said to be a C-topology relative to $\mathcal{U}$ if

(i) $H(X, \mathcal{U}) \subseteqq H(X, \mathcal{V})$,

(ii) $U \in \mathcal{U}$ if, and only if $U \cup V \in \mathcal{V}$ for every non-empty V in $\mathcal{V}$.

We have the following lemma. [1]