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LEFT INVERTIBILITY OF CLOSED OPERATORS MODULO AN α-COMPACT OPERATOR

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In a recent work, Peter Fillmore, Joseph Stampfli and James Williams listed seven conditions which, for a closed linear operator with dense domain, are equivalent to left invertibility modulo a compact operator. Cf. Theorem 1.1 of [3]. (The equivalence of some of these conditions was first shown by F. Wolf [8].) In an independent work, written around the same time, the author, together with Gerald Edgar and Sa Ge Lee, obtained a theorem (Theorem 2.6 of [2]) listing conditions, for a bounded operator, which are equivalent to left invertibility modulo an α-compact operator. (Let $\mathscr{H}$ be a Hilbert space of dimension h and let α denote a cardinal, $\aleph_0 \leqq \alpha \leqq h$. An α-compact operator is any bounded operator which is contained in $\mathscr{J}_\alpha$, the closed two-sided ideal which is the norm closure of the ideal $\mathscr{J}_\alpha$ of bounded operators of rank less than α. Cf. Theorem 0 of [2]. The $\aleph_0$-compact operators are precisely the compact operators.)

The purpose of this paper is to present one comprehensive theorem, listing many conditions equivalent to left invertibility modulo an α-compact operator, which integrates the two theorems mentioned above, and generalizes both. The Fillmore, Stampfli and Williams result is generalized to obtain appropriately modified conditions which are equivalent to left invertibility modulo each of the closed two-sided ideals of the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators. The Edgar, Ernest, Lee result is generalized to unbounded operators (specifically closed operators with dense domain).

Such a theory, formulated so as to describe the phenomena uniformly for all the closed two-sided ideals of $\mathscr{B}(\mathscr{H})$, without preferential treatment of the smallest such ideal (the compact operators) appears to us to be a necessary step preliminary to any investigation of similar questions in an arbitrary von Neumann algebra, where the collection of closed two-sided ideals is still more complicated.

The first section is preliminary. We generalize the notion of approximate nullity introduced in [2], to closed operators with dense domain. (This is also a generalization of the notion of approximate nullity as used by T. Kato in [7].) In our next section we present our main theorem.