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GENERALIZED CENTRAL SPHERES AND THE NOTION OF SPHERES IN RIEMANNIAN GEOMETRY

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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In a euclidean space E^{n+1} an n-plane or an n-sphere of radius r may be characterized as an umbilical hypersurface with mean curvature equal to 0 or $1/r$. A similar characterization is possible for an n-plane or an n-sphere in a euclidean space E^{n+p} where $p > 1$, as shown by E. Cartan [1], p. 231. Indeed, it is possible to determine all umbilical submanifolds of dimension n in an $(n + p)$-dimensional space form $\tilde{M}$, which can be regarded as "n-planes" or "n-spheres" according to whether the mean curvature is 0 or not.

In an arbitrary Riemannian manifold $\tilde{M}$ of dimension $n + p$, a natural analogue of an n-plane is an n-dimensional totally geodesic submanifold (equivalently, umbilical submanifold with zero mean curvature). In terms of a geometric notion of the development of curves, Cartan [1], p. 116, characterizes such n-planes in $\tilde{M}$ as follows. Let M be an n-dimensional submanifold of $\tilde{M}$. For every point x of M and for every curve τ in M starting at x, the development τ^* of τ into the euclidean tangent space $T_x(\tilde{M})$ lies in the euclidean subspace $T_x(M)$ if and only if M is totally geodesic in $\tilde{M}$.

The purpose of the present paper is to show that a natural analogue of an n-sphere in an arbitrary Riemannian manifold M is an n-dimensional *umbilical submanifold with non-zero parallel mean curvature vector* by characterizing such a submanifold as follows: for every point x of M and for every curve τ in M starting at x, the development τ^* lies in an n-sphere in $T_x(\tilde{M})$. The situation can be further clarified by introducing a generalization of central sphere defined in [5], which is also a generalization of the notion of osculating circle for a space curve. Namely, for an n-dimensional submanifold M with non-zero mean curvature in an arbitrary Riemannian manifold $\tilde{M}$, we associate to each point x of M a certain n-sphere $S^n(x)$ in $T_x(\tilde{M})$ which we call the *central n-sphere* at x. For every curve τ in M from x to y, the affine parallel displace-

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