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SUBGROUP OF SOME LIE GROUP AS A RIEMANNIAN SUBMANIFOLD

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Let G be a connected Lie group such that $\mathrm{Ad}(G)$ is compact. Then G admits a (positive definite) Riemannian metric g which is bi-invariant (left and right invariant). A submanifold H of G is endowed with the induced Riemannian metric g' by means of g. We consider G and H in such a situation. The purpose of this paper is to prove the following theorem.

THEOREM. *Let G be a connected Lie group such that* $\mathrm{Ad}(G)$ *is compact. Then an abstract subgroup H of G is a Lie subgroup (of dimension* > 0) *if and only if H is a totally geodesic submanifold of G.*

This is applicable, of course, in the case where G is connected and compact.

We keep in mind on the following facts. Let ∇ (resp. ∇_X) denote the covariant differential (resp. derivative) with respect to the Riemannian connection on G induced from g, then $\nabla_X Y = (1/2)[X, Y]$. Any 1-parameter subgroup $a(t)$, $-\infty < t < +\infty$ is a geodesic in G and the canonical parameter t is an affine parameter on the geodesic. Conversely a geodesic through e (unit element of G) is contained in a 1-parameter subgroup of G.

PROOF OF THE THEOREM. The necessity is easily verified. Conversely assume that H is an abstract subgroup of G which is a totally geodesic submanifold of G. The identity injection $H \to G$ is denoted by f. G and H are metric spaces by means of the Riemannian metric g and the induced Riemannian metric g' respectively, whose distance functions are denoted by $d_G(x, y)$, $x, y \in G$ and $d_H(x, y)$, $x, y \in H$ respectively. The topology of G (resp. H) coincides with that given by the distance d_G (resp. d_H), which is denoted by $\tilde{\Sigma}$ (resp. Σ). In general, Σ is stronger than the induced topology from $\tilde{\Sigma}$.

Let p be an arbitrary point of H and let V be an arbitrary open set (with respect to Σ) containing $q = L_a p (\in H)$. There exists an open ball