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ON THE TANGENT SPHERE BUNDLE OF A 2-SPHERE

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Introduction. Let S^2 be the unit sphere in a Euclidean space E^3 with the induced metric g. Then, the set of all unit tangent vectors $T_1(S^2)$ with the natural topology is the total space of the tangent sphere bundle $p: T_1(S^2) \rightarrow S^2$. $T_1(S^2)$ has a natural Riemannian metric. In this paper, we prove first that $T_1(S^2)$ with this metric is isometric with the elliptic space of constant curvature 1/4 (Theorem 1). Then, we give two proofs of a theorem which characterizes each geodesic on $T_1(S^2)$ as a vector field along a circle in S^2 (Theorem 2 and § 4). Finally, we give a theorem on the set of tangent vectors of a one parameter family of circles, the set corresponds to a Clifford surface in $T_1(S^2)$ regarded as an elliptic space (Theorem 4).

1. $T_1(S^2)$ **as a Riemannian manifold.** First we shall show

LEMMA 1. *$T_1(S^2)$ is diffeomorphic with the real projective space P^3.*

PROOF. For $y \in T_1(S^2)$, we consider the unit vector $e_1(y)$ which issues from the center O of S^2 and ends at the point $p(y)$. Then, the map $\psi: T_1(S^2) \rightarrow SO(3)$ defined by $y \rightarrow (e_1(y), e_2(y), e_1(y) \times e_2(y))$, where $e_2(y) \equiv y$ and $\times$ means vector product in E^3, is a diffeomorphism. On the other hand, it is well known that $SO(3)$ is diffeomorphic with P^3 (cf. for example [3] p. 115). Hence, $T_1(S^2)$ is diffeomorphic with P^3.

Now, let U be an arbitrary coordinate neighborhood with local coordinates $x^a (a, b, c = 1, 2)$ and y^a be components of a tangent vector y in U with respect to the natural frame $\partial/\partial x^a$. Then, $p^{-1}(U)$ gives a coordinate neighborhood of $T_1(S^2)$ with local coordinates (x^a, y^a). By virtue of the induced metric g on S^2 in E^3, the natural Riemannian metric $\hat{g}$ on $T_1(S^2)$ is given by the following line element:

$$d\sigma^2 = g_{bc}(x)dx^b dx^c + g_{bc}(x)\delta y^b \delta y^c \,, \tag{1.1}$$

([2]) where we have put

$$g_{bc}(x)y^b y^c = 1 \,, \qquad \delta y^b = dy^b + \left\{ {b \atop ef} \right\} y^e dx^f \,. \tag{1.2}$$

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