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MARTINGALE SEQUENCE OF BOUNDED VARIATION

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Introduction. D. L. Burkholder proved ([1] Theorem 5) that an L^1-bounded martingale sequence is of bounded variation a.s. on every atom of the basic probability space. This result was proved by the general convergence theorem of martingale transforms. We shall give in § 1 a direct simple proof of this theorem. In § 2 we shall give some counter examples concerning the majoration inequalities. And in § 3 we shall show that the conclusion of the Burkholder theorem is not necessarily true on the atomless part of the probability space.

1. THEOREM (Burkholder). *Let* $X = \{X_n, \mathscr{F}_n, n \geqq 1\}$ *be a martingale defined on a probability space* $(\Omega, \mathscr{F}, \mathrm{P})$, *and denote its difference sequence by* $\{d_n\}$, $d_n = X_n - X_{n-1}$, $n = 1, 2, \cdots$; $X_0 = 0$. *If* X *is* L^1 *bounded*:

$$\sup_n \mathrm{E}\,|X_n| = K < \infty ,$$

and if A *is an atom of the probability space, then* $\sum_n |d_n| < \infty$, *a.s. on* A.

PROOF. Every random variable is constant a.s. on every atom, so we can put $X_n = a_n$, a.s. $(n = 1, 2, \cdots)$ on the atom A where $\mathrm{P}(A) > 0$ and a_n are real constants. The sequence $\{a_n\}$ is bounded, say $|a_n| \leqq a$ for all n, as we see easily from the inequalities:

$$|a_n|\,\mathrm{P}(A) \leqq \mathrm{E}\,|X_n| \leqq \sup_n \mathrm{E}\,|X_n| = K .$$

Put $A_n = \{X_n = a_n\}$ and $A_n^* = \bigcap_{j=1}^n A_j$. Clearly $A_n, A_n^* \in \mathscr{F}_n$, $A_n^* \supset A_{n+1}^*$ and $A_n^* \supset A$ for all n, since $A_n \in \mathscr{F}_n$ and $A_n \supset A$. By the martingale equality

$$\int_{A_n^*} a_n = \int_{A_n^*} X_n = \int_{A_n^*} X_{n+1} = \int_{A_{n+1}^*} a_{n+1} + \int_{A_n^* - A_{n+1}^*} X_{n+1} ,$$

hence easily

$$(a_n - a_{n+1})\mathrm{P}(A_{n+1}^*) = \int_{A_n^* - A_{n+1}^*} (X_{n+1} - a_n) .$$

So that it follows from the submartingale inequality applied to $\{|X_n|,$