Tôhoku Math. Journ.
28 (1976), 469-477.

HOMOGENEOUS RIEMANNIAN MANIFOLDS COVERED BY $S^m \times S^n$

HITOSHI TAKAGI

(Received October 23, 1975)

1. Introduction. The purpose of the present paper is to classify the homogeneous Riemannian manifolds covered by a product of two ordinary spheres with standard metrics. This classification reduces to classify the finite subgroups of the direct product group $\boldsymbol{Q}' \times \boldsymbol{Q}'$, where $\boldsymbol{Q}'$ is the multicative group of unit quaternions.

For this purpose, we shall use the classification of the homogeneous Riemannian manifolds of positive constant curvature by J. A. Wolf (cf. [3]) and refer to the author's papers [1] and [2].

2. Clifford translations of $S^m \times S^n$. Let $S^m(K)$ $(m \geqq 2)$ denote an m-dimensional sphere of radius $K^{-1/2}$ in an $(m+1)$-dimensional Euclidean space. $S^m(K)$ is viewed as a set of vectors of norm $K^{-1/2}$ in a Euclidean vector space $\boldsymbol{R}^{m+1}$. Then the group $O(m+1)$ of all linear isometries of $\boldsymbol{R}^{m+1}$ is considered as the group of all isometries of $S^m(K)$. The Riemannian product of $S^m(K)$ and $S^n(K')$ $(m, n \geqq 2)$ is given by $S^m(K) \times S^n(K') = \{(x, y) \in \boldsymbol{R}^{m+1} \times \boldsymbol{R}^{n+1} = \boldsymbol{R}^{m+n+2}; \|x\| = K^{-1/2}, \|y\| = (K')^{-1/2}\}$. Then the direct product $O(m+1) \times O(n+1)$ is a group of isometries of $S^m(K) \times S^n(K')$, which is considered as a subgroup of $O(m+n+2)$. On the other hand, the linear isometry s of $\boldsymbol{R}^{m+1} \times \boldsymbol{R}^{m+1}$ defined by $s(x, y) = (y, x)$ is considered as an isometry of $S^m(K) \times S^m(K)$. $(O(m+1) \times O(m+1)) \cup (O(m+1) \times O(m+1))s$ or $O(m+1) \times O(n+1)$ is the group of all isometries of $S^m(K) \times S^n(K')$, according as $S^m(K)$ and $S^n(K')$ are isometric or not (cf. p. 243 [3]).

Let $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ denote the algebra of real quaternions and the multicative group of unit quaternions respectively. That is, $\boldsymbol{Q}' = \{a = a_1 + a_2 i + a_3 j + a_4 k;\ i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j, a_t \in \boldsymbol{R}\ (t = 1, 2, 3, 4), \sum_{t=1}^{4} a_t^2 = 1\}$. Then, $\boldsymbol{Q}'$ contains $\boldsymbol{C}' = \{a = a_1 + a_2 i;\ a_1, a_2 \in \boldsymbol{R}, a_1^2 + a_2^2 = 1\}$ as a subgroup. $\boldsymbol{C}'$ contains $\boldsymbol{Z}_2 = \{\pm 1\}$ as a subgroup. By certain representations, $\boldsymbol{Q}'$, $\boldsymbol{C}'$ and $\boldsymbol{Z}_2$ are considered as closed subgroups $O(4l)$, $O(2l)$ and $O(l)$ $(l = 1, 2, \cdots)$ respectively (cf. [1], [2]). Then, $\boldsymbol{Q}'$, $\boldsymbol{C}'$ and $\boldsymbol{Z}_2$ are considered as groups of isometries of S^{4l-1}, S^{2l-1} and S^{l-1} respectively.