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CURVATURE IDENTITIES FOR HERMITIAN AND ALMOST HERMITIAN MANIFOLDS

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1. Introduction. Among Riemannian manifolds Kähler manifolds have an especially rich geometric and topological structure. Perhaps the main reason for this is that the curvature tensor of a Kähler manifold satisfies a special identity, the Kähler identity $R_{WXJYJZ} = R_{WXYZ}$. From this identity it is possible to prove many interesting theorems about harmonic forms, for example.

Because of the strength of the Kähler identity, the question naturally arises: what sort of identities, if any, exist for more general types of almost Hermitian manifolds. In this paper we show that indeed curvature identities exist for the classes $\mathscr{H}$ and $\mathscr{QK}$ of Hermitian and quasi-Kähler manifolds.

That a curvature identity for Hermitian manifolds has not been found or made use of up to now in the literature is somewhat surprising. The defining property of the class $\mathscr{H}$ is that

$$[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0 .$$

We shall show that

$$[R_{XY}, J] + J[R_{JXY}, J] + J[R_{XJY}, J] - [R_{JXJY}, J] = 0$$

holds for $\mathscr{H}$. Since the structure of Hermitian manifolds is almost as rich as that of Kähler manifolds, this identity should prove productive.

Quasi-Kähler manifolds are important because they include the classes $\mathscr{AK}$ and $\mathscr{NK}$ of almost Kähler and nearly Kähler manifolds. A large part of the theory of the geometry and topology of Kähler manifolds can be carried over to the class $\mathscr{NK}$ [8]. However, not very much is known about $\mathscr{AK}$. We give a new curvature identity in Corollary (4.3). Hervella and Naveira [12] have derived a special case of our curvature identity for the class $\mathscr{QK}$. They also give several geometric applications.

In § 2 we explain our notation and prove an important lemma. We study curvature identities of Hermitian manifolds in § 3. Then § 4 is devoted to curvature identities in quasi-Kähler and almost Kähler mani-